Pointer Concurrent Games and the Resource Calculus

Jeux concurrents à pointeurs et calcul à ressources

Lison Blondeau-Patissier

Aix-Marseille Université

PhD Defence, 04/12/2025

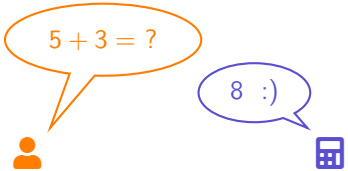
Supervised by Pierre Clairambault and Lionel Vaux Auclair

Sémantique des langages de programmation

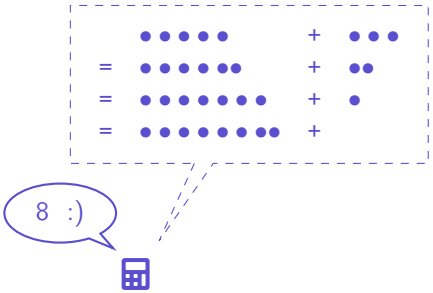
La sémantique sauve des vies. — Lionel Vaux Auclair

Sémantique des langages de programmation

La sémantique sauve des vies. — Lionel Vaux Auclair

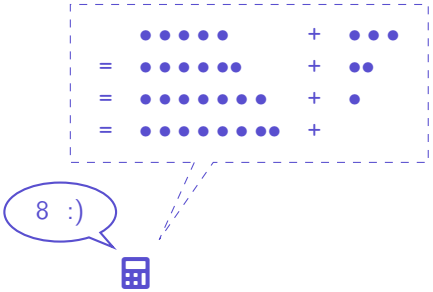


Sémantiques des langages de programmation

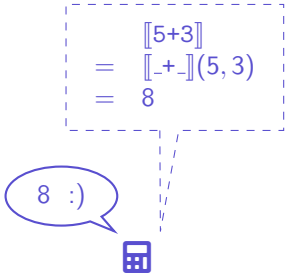


Sémantique *opérationnelle*

Sémantiques des langages de programmation

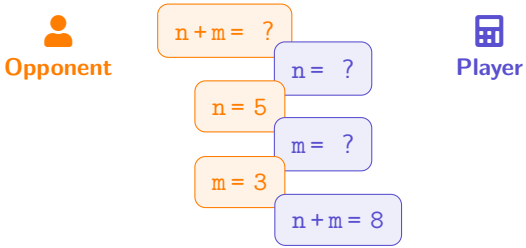


Sémantique *opérationnelle*

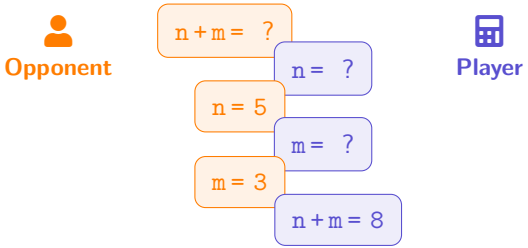


Sémantique *dénotationnelle*

Sémantique des jeux

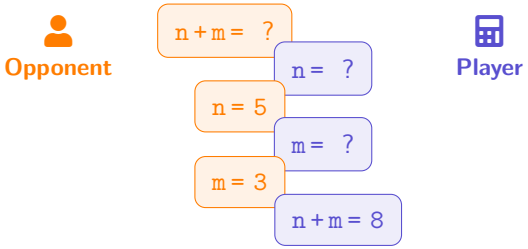


Sémantique des jeux



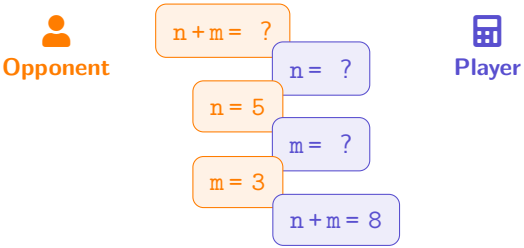
Jeux	Programmation
Parties	Exécutions 5+3=8

Sémantique des jeux



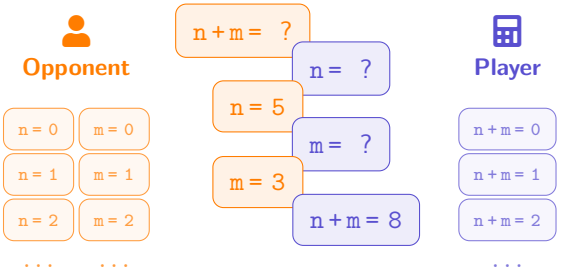
Jeux	Programmation
Parties	Exécutions 5+3=8
Stratégies	Programmes Addition

Sémantique des jeux



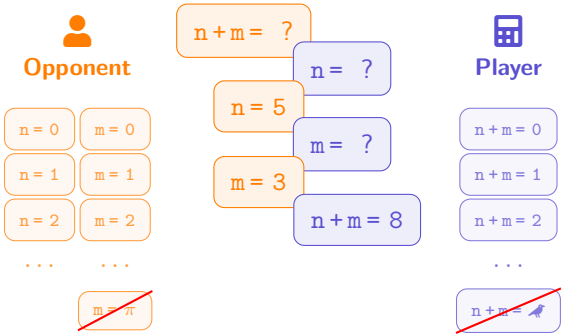
Jeux	Programmation
	Types (nat,nat) → nat
Parties	Exécutions 5+3=8
Stratégies	Programmes Addition

Sémantique des jeux



Jeux	Programmation
Arènes	Types (nat,nat) → nat
Parties	Exécutions 5+3=8
Stratégies	Programmes Addition

Sémantique des jeux



Jeux	Programmation
Arènes	Types (nat,nat) → nat
Parties	Exécutions 5+3=8
Stratégies	Programmes Addition

Calculs

Fig 1: Des vaches.

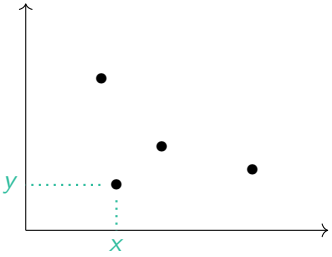


Fig 2: Aussi des vaches ?

Le λ-calcul

Les termes du λ-calcul sont de la forme :

$$
\begin{array}{lcll}
M, N, L, \ldots & ::= & x & \textit{(variable)} \\
& | & \lambda x.M & \textit{(abstraction)} \\
& | & M\ N & \textit{(application)}
\end{array}
$$

Le λ-calcul

Les termes du λ-calcul sont de la forme :

$$
\begin{array}{lcll}
M, N, L, \ldots & ::= & x & & \textit{(variable)} \\
& | & \lambda x.M & \text{“}x \mapsto M\text{”} & \textit{(abstraction)} \\
& | & M\ N & \text{“}M(N)\text{”} & \textit{(application)}
\end{array}
$$

Le λ-calcul

Les termes du λ-calcul sont de la forme :

$$
\begin{array}{lcll}
M, N, L, \ldots & ::= & x & & \textit{(variable)} \\
& | & \lambda x.M & \text{“}x \mapsto M\text{”} & \textit{(abstraction)} \\
& | & M\ N & \text{“}M(N)\text{”} & \textit{(application)}
\end{array}
$$

Exécution d'un programme :

```
def double(n):
    return n + n
```

Le λ-calcul

Les termes du λ-calcul sont de la forme :

$$
\begin{array}{lcll}
M, N, L, \ldots & ::= & x & & \textit{(variable)} \\
& | & \lambda x.M & \text{“}x \mapsto M\text{”} & \textit{(abstraction)} \\
& | & M\ N & \text{“}M(N)\text{”} & \textit{(application)}
\end{array}
$$

Exécution d'un programme :

```
def double(n):
    return n + n
```

`double(3)` $\rightarrow$ `3 + 3` $\rightarrow$ `6`

Le λ-calcul

Les termes du λ-calcul sont de la forme :

$$
\begin{array}{lcll}
M, N, L, \ldots & ::= & x & & \textit{(variable)} \\
& | & \lambda x.M & \text{“}x \mapsto M\text{”} & \textit{(abstraction)} \\
& | & M\ N & \text{“}M(N)\text{”} & \textit{(application)}
\end{array}
$$

Exécution d'un programme :

```
def double(n):
    return n + n
```

`double(3)` $\rightarrow$ `3 + 3` $\rightarrow$ `6`

Opération de β-réduction :

$$(\lambda x.M)\ N \ \rightarrow_\beta \ M[N/x]$$

Le λ-calcul

Les termes du λ-calcul sont de la forme :

$$
\begin{array}{rcll}
M, N, L, \ldots & ::= & x & & \textit{(variable)} \\
& | & \lambda x.M & \text{“}x \mapsto M\text{”} & \textit{(abstraction)} \\
& | & M\ N & \text{“}M(N)\text{”} & \textit{(application)}
\end{array}
$$

Exécution d'un programme :

```
def double(n):
    return n + n
```

`double(3)` $\rightarrow$ `3 + 3` $\rightarrow$ `6`

Opération de β-réduction :

$$(\lambda x.M)\ N \ \rightarrow_\beta \ M[N/x]$$

Exemple. $(\lambda x.y\,x\,x)\ N \rightarrow_\beta y\,N\,N.$

Le λ-calcul typé

Jeux	Calculs	Programmation
Arènes		Types
Parties		Exécutions
Stratégies	Termes	Programmes

Le λ-calcul typé

Jeux	Calculs	Programmation
Arènes	Types	Types
Parties		Exécutions
Stratégies	Termes	Programmes

Le λ-calcul typé

Les types du λ-calcul sont :

$$A, B, \ldots \quad ::= \quad \alpha \qquad \textit{(type de base)}$$
$$\qquad\qquad\quad \mid \; A \to B \qquad \textit{(fonction)}$$

Jeux	Calculs	Programmation
Arènes	Types	Types
Parties		Exécutions
Stratégies	Termes	Programmes

Le λ-calcul typé

Les types du λ-calcul sont :

$$A, B, \ldots \quad ::= \quad \alpha \qquad \textit{(type de base)}$$
$$\mid \quad A \to B \qquad \textit{(fonction)}$$

Jeux	Calculs	Programmation
Arènes	Types	Types
Parties		Exécutions
Stratégies	Termes	Programmes

Exemple.

$$\lambda x.x : \alpha \to \alpha$$

Le λ-calcul typé

Les types du λ-calcul sont :

$$A, B, \ldots \quad ::= \quad \alpha \qquad \textit{(type de base)}$$
$$\mid \; A \to B \qquad \textit{(fonction)}$$

Jeux	Calculs	Programmation
Arènes	Types	Types
Parties		Exécutions
Stratégies	Termes	Programmes

Exemple.

$$\lambda x.x : \alpha \to \alpha \qquad \lambda f.\ (\lambda x.\ f\ (f\ x)) : (\alpha \to \alpha) \to (\alpha \to \alpha)$$

Le λ-calcul typé

Les types du λ-calcul sont :

$$A, B, \ldots \quad ::= \quad \alpha \qquad \textit{(type de base)}$$
$$\mid \; A \to B \qquad \textit{(fonction)}$$

Jeux	Calculs	Programmation
Arènes	Types	Types
Parties		Exécutions
Stratégies	Termes	Programmes

Exemple.

$$\lambda x.x : \alpha \to \alpha \qquad \lambda f.\ (\lambda x.\ f\ (f\ x)) : (\alpha \to \alpha) \to (\alpha \to \alpha)$$

Le calcul à ressources

Les termes du calcul à ressources sont de la forme :

$$
\begin{array}{lcll}
s, t, u \ldots & ::= & x & \textit{(variable)} \\
& | & \lambda x.s & \textit{(abstraction)} \\
& | & s\ [t_1, t_2, \ldots, t_n] & \textit{(application)}
\end{array}
$$

Le calcul à ressources

Les termes du calcul à ressources sont de la forme :

$$
\begin{array}{lcll}
s, t, u \ldots & ::= & x & \textit{(variable)} \\
& | & \lambda x.s & \textit{(abstraction)} \\
& | & s\ [t_1, t_2, \ldots, t_n] & \textit{(application)}
\end{array}
$$

Exemple.

$$(\lambda x.y\,[x]\,[x])\ \ [t_1, t_2] \qquad \to_\beta \quad y\,[t_1]\,[t_2]\ +\ y\,[t_2]\,[t_1]$$

Le calcul à ressources

Les termes du calcul à ressources sont de la forme :

$$
\begin{array}{lcll}
s, t, u \ldots & ::= & x & \textit{(variable)} \\
& | & \lambda x.s & \textit{(abstraction)} \\
& | & s\ [t_1, t_2, \ldots, t_n] & \textit{(application)}
\end{array}
$$

Exemple.

$$(\lambda x.y\,[x]\,[x])\ [t_1, t_2] \quad \to_\beta \quad y\,[t_1]\,[t_2] \ + \ y\,[t_2]\,[t_1]$$

$$(\lambda x.y\,[x]\,[x])\ [t_1] \quad \to_\beta \quad 0$$

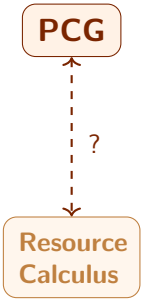
$$(\lambda x.y\,[x]\,[x])\ [t_1, t_2, t_3] \quad \to_\beta \quad 0$$

Pointer Concurrent Games and the Resource Calculus

Games	Calculi	Programming
Arenas	Types	Types
Plays	Resource terms	Executions
Strategies	Terms	Programs

Pointer Concurrent Games and the Resource Calculus

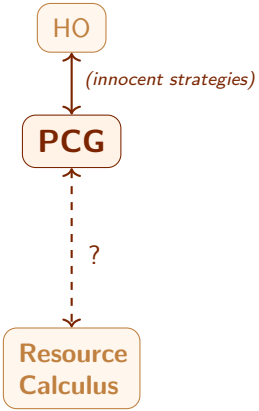
This thesis:



Games	Calculi	Programming
Arenas	Types	Types
Plays	Resource terms	Executions
Strategies	Terms	Programs

Pointer Concurrent Games and the Resource Calculus

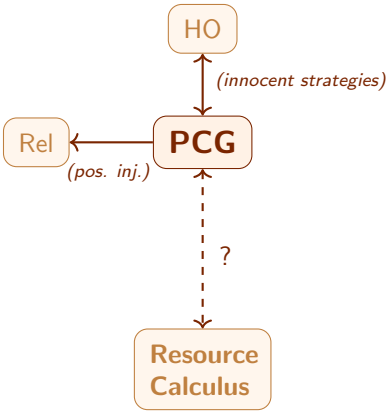
This thesis:



Games	Calculi	Programming
Arenas	Types	Types
Plays	Resource terms	Executions
Strategies	Terms	Programs

Pointer Concurrent Games and the Resource Calculus

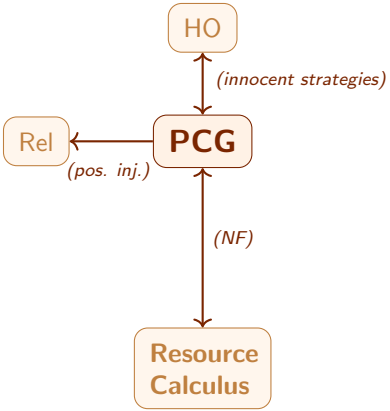
This thesis:



Games	Calculi	Programming
Arenas	Types	Types
Plays	Resource terms	Executions
Strategies	Terms	Programs

Pointer Concurrent Games and the Resource Calculus

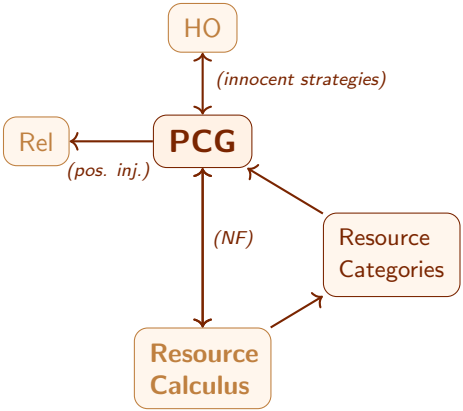
This thesis:



Games	Calculi	Programming
Arenas	Types	Types
Plays	Resource terms	Executions
Strategies	Terms	Programs

Pointer Concurrent Games and the Resource Calculus

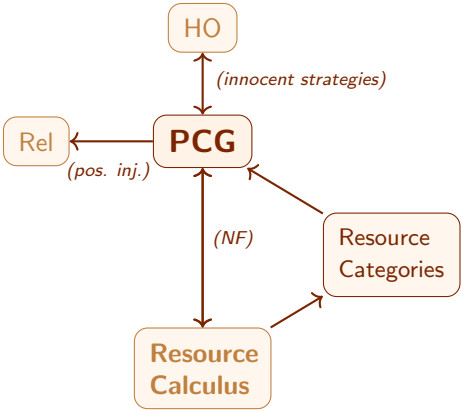
This thesis:



Games	Calculi	Programming
Arenas	Types	Types
Plays	Resource terms	Executions
Strategies	Terms	Programs

Pointer Concurrent Games and the Resource Calculus

This thesis:



Games	Calculi	Programming
Arenas	Types	Types
Plays	Resource terms	Executions
Strategies	Terms	Programs

This talk:

1. Introduction to PCG (and HO)
2. Isomorphism for terms in normal form
3. Dynamics of PCG

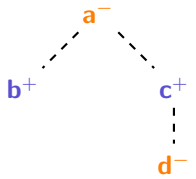
I – Introduction to Game Semantics

- Hyland-Ong Games
- Pointer Concurrent Games

Arenas (Types)

An **arena** is a forest $\mathsf{A} = \langle |\mathsf{A}|, \to_{\mathsf{A}}, \mathrm{pol}_{\mathsf{A}} \rangle$ with:

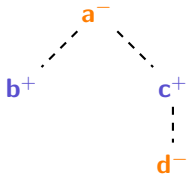
- $\mathrm{pol}_{\mathsf{A}} : |\mathsf{A}| \to \{-, +\}$
- $\to_{\mathsf{A}}$ alternating.



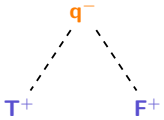
Arenas (Types)

An **arena** is a forest $\mathsf{A} = \langle |\mathsf{A}|, \to_{\mathsf{A}}, \mathrm{pol}_{\mathsf{A}} \rangle$ with:

- $\mathrm{pol}_{\mathsf{A}} : |\mathsf{A}| \to \{-, +\}$
- $\to_{\mathsf{A}}$ alternating.



Examples.

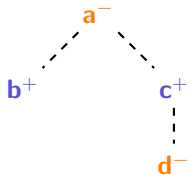


Arena **bool**

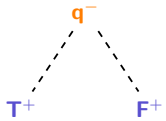
Arenas (Types)

An **arena** is a forest $A = \langle |A|, \rightarrow_A, \mathrm{pol}_A \rangle$ with:

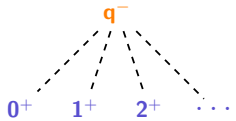
- $\mathrm{pol}_A : |A| \to \{-, +\}$
- $\rightarrow_A$ alternating.



Examples.



Arena **bool**

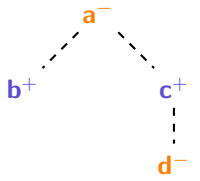


Arena **nat**

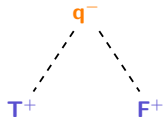
Arenas (Types)

An **arena** is a forest $\mathsf{A} = \langle |\mathsf{A}|, \rightarrow_{\mathsf{A}}, \mathrm{pol}_{\mathsf{A}} \rangle$ with:

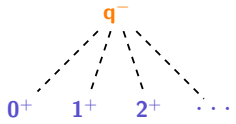
- $\mathrm{pol}_{\mathsf{A}} : |\mathsf{A}| \to \{-, +\}$
- $\rightarrow_{\mathsf{A}}$ alternating.



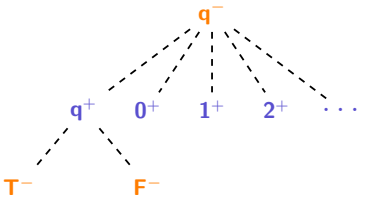
Examples.



Arena **bool**



Arena **nat**

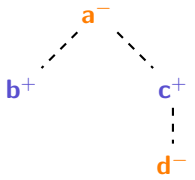


Arena **bool** $\to$ **nat**

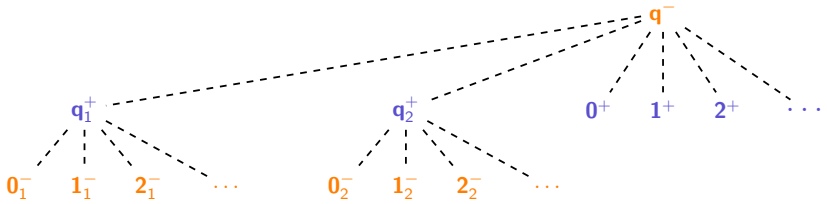
Arenas (Types)

An **arena** is a forest $\mathsf{A} = \langle |\mathsf{A}|, \to_{\mathsf{A}}, \mathrm{pol}_{\mathsf{A}} \rangle$ with:

- $\mathrm{pol}_{\mathsf{A}} : |\mathsf{A}| \to \{-, +\}$
- $\to_{\mathsf{A}}$ alternating.



Examples.



Arena $\mathbf{nat}_1 \to \mathbf{nat}_2 \to \mathbf{nat}$

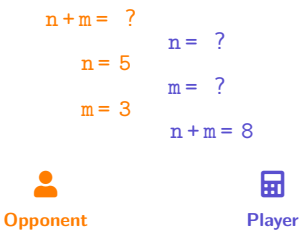
Plays (Executions)

A **play** s in an arena A is an *alternating* sequence of events on A, with *pointers*.

Plays (Executions)

A **play** s in an arena A is an *alternating* sequence of events on A, with *pointers*.

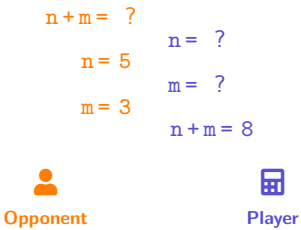
Example.

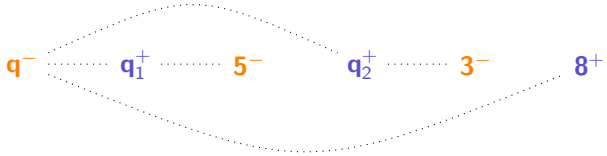


Plays (Executions)

A **play** s in an arena A is an *alternating* sequence of events on A, with *pointers*.

Example.



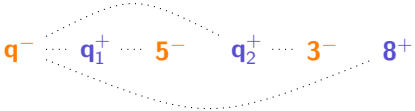


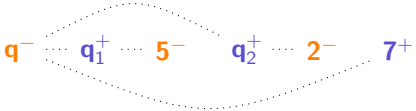
A play in $\mathbf{nat}_1 \to \mathbf{nat}_2 \to \mathbf{nat}$

Strategies (Programs)

A **strategy** is a set of plays *(that is deterministic, prefix closed...)*.

Example. The strategy for  includes:

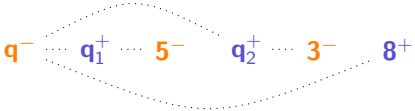


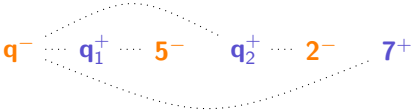


Strategies (Programs)

A **strategy** is a set of plays *(that is deterministic, prefix closed...)*.

Example. The strategy for  includes:

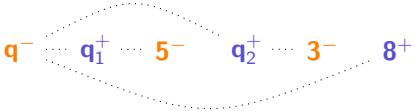




Strategies (Programs)

A **strategy** is a set of plays *(that is deterministic, prefix closed...)*.

Example. The strategy for  includes:



Innocent Strategies

Intuition: 🖩 should forget when Opponent backtracks.

Innocent Strategies

Intuition:  should forget when Opponent backtracks.

Melliès' **homotopy relation** $\sim$ equates plays which only differ by Opponent's scheduling.

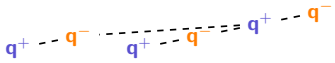
Innocent Strategies

Intuition: 🖩 should forget when Opponent backtracks.

Melliès' **homotopy relation** $\sim$ equates plays which only differ by Opponent's scheduling.

Example.

$$((\alpha \to \alpha) \to (\alpha \to \alpha) \to \alpha) \to \alpha$$



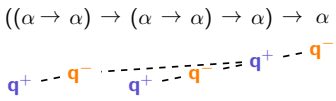
$$\lambda f^{(\alpha\to\alpha)\to(\alpha\to\alpha)\to\alpha}.\ f\ [\lambda x^{\alpha}.\ x,\ \lambda y^{\alpha}.\ y]\ [\lambda z^{\alpha}.\ f\ [\,]\ [\,]]$$

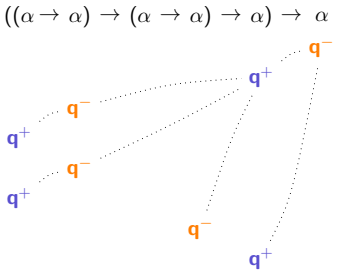
Innocent Strategies

Intuition: 🖩 should forget when Opponent backtracks.

Melliès' **homotopy relation** $\sim$ equates plays which only differ by Opponent's scheduling.

Example.





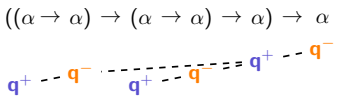
$\lambda f^{(\alpha\to\alpha)\to(\alpha\to\alpha)\to\alpha}.\ f\ [\lambda x^{\alpha}.\ x,\ \lambda y^{\alpha}.\ y]\ [\lambda z^{\alpha}.\ f\ [\,]\ [\,]]$

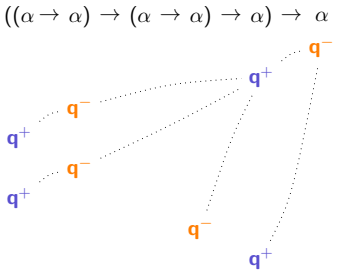
Innocent Strategies

Intuition: 🖩 should forget when Opponent backtracks.

Melliès' **homotopy relation** $\sim$ equates plays which only differ by Opponent's scheduling.

Example.





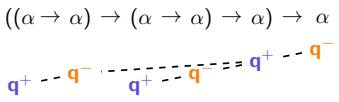
$\lambda f^{(\alpha\to\alpha)\to(\alpha\to\alpha)\to\alpha}.\ f\ [\lambda x^{\alpha}.\ x,\ \lambda y^{\alpha}.\ y]\ [\lambda z^{\alpha}.\ f\ [\,]\ [\,]]$

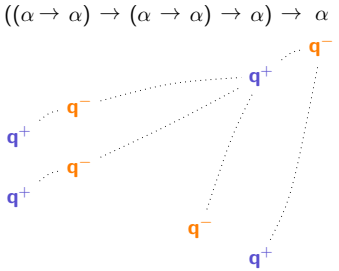
Innocent Strategies

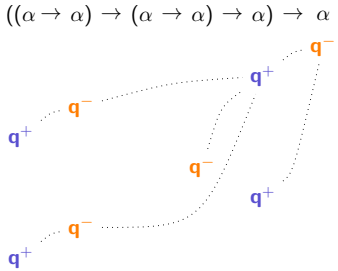
Intuition: 🖩 should forget when Opponent backtracks.

Melliès' **homotopy relation** $\sim$ equates plays which only differ by Opponent's scheduling.

Example.







$\lambda f^{(\alpha\to\alpha)\to(\alpha\to\alpha)\to\alpha}.\ f\ [\lambda x^{\alpha}.\ x,\ \lambda y^{\alpha}.\ y]\ [\lambda z^{\alpha}.\ f\ [\,]\ [\,]]$

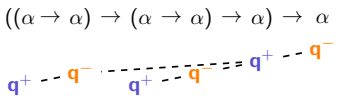
Innocent Strategies

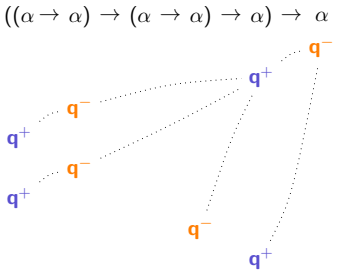
Intuition: 🖩 should forget when Opponent backtracks.

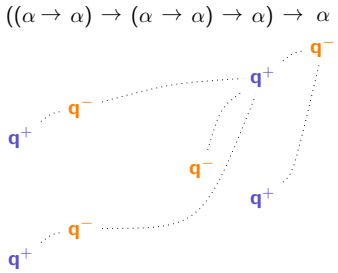
Melliès' **homotopy relation** $\sim$ equates plays which only differ by Opponent's scheduling.

Example.

Innocent strategies are stable by $\sim$.







$$\lambda f^{(\alpha\to\alpha)\to(\alpha\to\alpha)\to\alpha}.\ f\ [\lambda x^{\alpha}.\ x,\ \lambda y^{\alpha}.\ y]\ [\lambda z^{\alpha}.\ f\ [\,]\ [\,]]$$

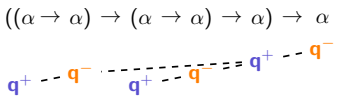
Innocent Strategies

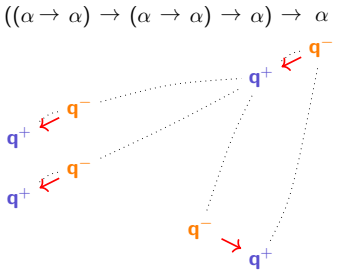
Intuition: 🖩 should forget when Opponent backtracks.

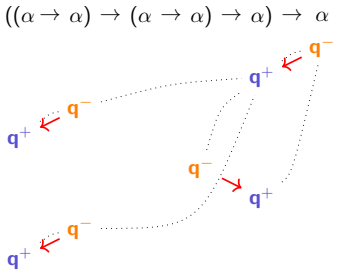
Melliès' **homotopy relation** $\sim$ equates plays which only differ by Opponent's scheduling.

Example.

Innocent strategies are stable by $\sim$.





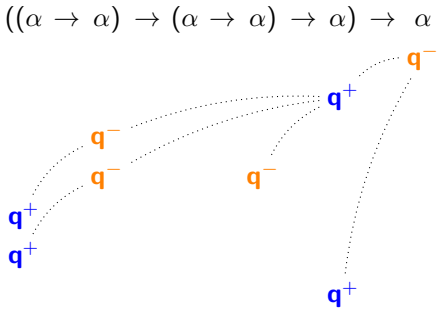


$\lambda f^{(\alpha\to\alpha)\to(\alpha\to\alpha)\to\alpha}.\ f\ [\lambda x^{\alpha}.\ x,\ \lambda y^{\alpha}.\ y]\ [\lambda z^{\alpha}.\ f\ [\,]\ [\,]]$

Pointer Concurrent Games: Configurations and Augmentations

A **configuration** is $x = \langle |x|, \rightarrow_x, \partial_x \rangle$ where:

- $\langle |x|, \rightarrow_x \rangle$ is a finite forest
- $\partial_x : |x| \rightarrow |A|$ preserves minimality and causality



Pointer Concurrent Games: Configurations and Augmentations

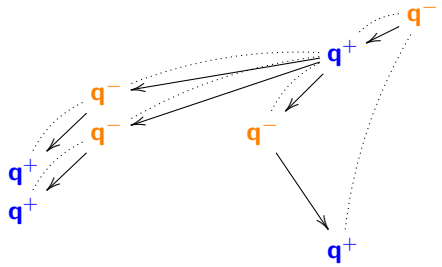
A **configuration** is $x = \langle |x|, \rightarrow_x, \partial_x \rangle$ where:

- $\langle |x|, \rightarrow_x \rangle$ is a finite forest
- $\partial_x : |x| \rightarrow |A|$ preserves minimality and causality

An **augmentation** is $p = \langle (\!|p|\!), \rightarrow_p \rangle$ where:

- $(\!|p|\!)$ is a configuration,
- $\langle |p|, \rightarrow_p \rangle$ is a forest which *augments* the static links in $\rightarrow_{(\!|p|\!)}$ with **causal links** from negative to positive moves.

$$((\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha$$



$$\lambda f^{(\alpha\rightarrow\alpha)\rightarrow(\alpha\rightarrow\alpha)\rightarrow\alpha}.f\,[\lambda x^{\alpha}.x,\ \lambda y^{\alpha}.y]\,[\lambda z^{\alpha}.f\,[\,]\,[\,]\,]$$

II – Static isomorphism between PCG and the Resource Calculus

- The extensional resource calculus
- Augmentations are normal resource terms

Types and Contexts

Types: $A, B, \ldots ::= \alpha \mid A \to B$.

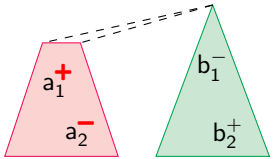
Types and Contexts

Types: $A, B, \ldots ::= \alpha \mid A \to B$.

Interpretation:

$$[\![\alpha]\!] \stackrel{\text{def}}{=} \mathsf{o}$$

$$[\![A \to B]\!] \stackrel{\text{def}}{=} [\![A]\!] \Rightarrow [\![B]\!]$$



Arena A ⇒ B.

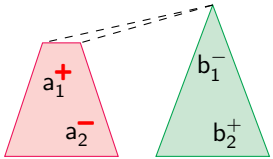
Types and Contexts

Types: $A, B, \ldots ::= \alpha \mid A \to B.$

Interpretation:

$$[\![\alpha]\!] \stackrel{\text{def}}{=} \mathsf{o}$$

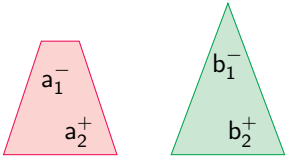
$$[\![A \to B]\!] \stackrel{\text{def}}{=} [\![A]\!] \Rightarrow [\![B]\!]$$



Arena A ⇒ B.

Contexts:

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \bigotimes_{(x:A)\in\Gamma} [\![A]\!]$$



Arena A ⊗ B.

How do we type resource terms?

$$\frac{}{\Gamma, x : A \vdash x : A}\ (\text{var}) \qquad \frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x.s : A \to B}\ (\text{abs})$$

$$\frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash t_1 : A \qquad \dots \qquad \Gamma \vdash t_n : A}{\Gamma \vdash s\ [t_1, \dots, t_n] : B}\ (\text{app})$$

How do we type resource terms?

$$\frac{}{\Gamma, x : A \vdash x : A}\ (\text{var}) \qquad \frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x.s : A \to B}\ (\text{abs})$$

$$\frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash \bar{t} : A}{\Gamma \vdash s\ \bar{t} : B}\ (\text{app}) \qquad \frac{\Gamma \vdash t_1 : A \qquad \dots \qquad \Gamma \vdash t_n : A}{\Gamma \vdash \bar{t} : A}\ (\text{bag})$$

The *extensional* Resource Calculus

$$\frac{\Gamma, x : A \vdash_{\mathsf{Tm}} s : B}{\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B}\ (\mathrm{abs}) \qquad \frac{\Gamma \vdash_{\mathsf{Tm}} s_1 : A \quad \cdots \quad \Gamma \vdash_{\mathsf{Tm}} s_n : A}{\Gamma \vdash_{\mathsf{Bg}} [s_1, \ldots, s_n] : A}\ (\mathrm{bag})$$

$$\frac{\Gamma \vdash_{\mathsf{Tm}} s : A \to B \qquad \Gamma \vdash_{\mathsf{Bg}} \bar{t} : A}{\Gamma \vdash_{\mathsf{Tm}} s\ \bar{t} : B}\ (\mathrm{app})$$

Typing rules for the simply-typed η-long resource calculus.

The *extensional* Resource Calculus

$$\frac{\Gamma, x : A \vdash_{\mathsf{Tm}} s : B}{\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B}\ (\text{abs}) \qquad \frac{\Gamma \vdash_{\mathsf{Tm}} s_1 : A \quad \cdots \quad \Gamma \vdash_{\mathsf{Tm}} s_n : A}{\Gamma \vdash_{\mathsf{Bg}} [s_1, \ldots, s_n] : A}\ (\text{bag})$$

$$\frac{\Gamma \vdash_{\mathsf{Tm}} s : A \to B \qquad \Gamma \vdash_{\mathsf{Bg}} \bar{t} : A}{\Gamma \vdash_{\mathsf{Tm}} s\ \bar{t} : B}\ (\text{app})$$

$$\frac{\Gamma, x : \ldots \vdash_{\mathsf{Bg}} \bar{s}_1 : A_1 \quad \cdots \quad \Gamma, x : \ldots \vdash_{\mathsf{Bg}} \bar{s}_n : A_n}{\Gamma, x : A_1 \to \ldots \to A_n \to \alpha \vdash_{\mathsf{Tm}} x\ \bar{s}_1\ \ldots\ \bar{s}_2 : \alpha}\ (\text{var})$$

Typing rules for the simply-typed η-long resource calculus.

The *extensional* Resource Calculus

$$\frac{\Gamma, x : A \vdash_{\mathsf{Tm}} s : B}{\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B}\ (\text{abs}) \qquad \frac{\Gamma \vdash_{\mathsf{Tm}} s_1 : A \quad \cdots \quad \Gamma \vdash_{\mathsf{Tm}} s_n : A}{\Gamma \vdash_{\mathsf{Bg}} [s_1, \ldots, s_n] : A}\ (\text{bag})$$

$$\frac{\Gamma \vdash_{\mathsf{Tm}} s : A \to B \qquad \Gamma \vdash_{\mathsf{Bg}} \bar{t} : A}{\Gamma \vdash_{\mathsf{Tm}} s\ \bar{t} : B}\ (\text{app})$$

$$\frac{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Sq}} \vec{s} : \vec{A}}{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Tm}} x\ \vec{s} : \alpha}\ (\text{var}) \qquad \frac{\Gamma \vdash_{\mathsf{Bg}} \bar{s}_1 : A_1 \quad \cdots \quad \Gamma \vdash_{\mathsf{Bg}} \bar{s}_n : A_n}{\Gamma \vdash_{\mathsf{Sq}} \langle \bar{s}_1, \ldots, \bar{s}_n \rangle : \langle A_1, \ldots, A_n \rangle}\ (\text{seq})$$

Typing rules for the simply-typed η-long resource calculus.

The *extensional* Resource Calculus

$$\frac{\Gamma, x : A \vdash_{\mathsf{Tm}} s : B}{\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B}\ (\text{abs}) \qquad \frac{\Gamma \vdash_{\mathsf{Tm}} s_1 : A \quad \cdots \quad \Gamma \vdash_{\mathsf{Tm}} s_n : A}{\Gamma \vdash_{\mathsf{Bg}} [s_1, \ldots, s_n] : A}\ (\text{bag})$$

$$\frac{\Gamma \vdash_{\mathsf{Tm}} s : A \to B \qquad \Gamma \vdash_{\mathsf{Bg}} \bar{t} : A}{\Gamma \vdash_{\mathsf{Tm}} s\ \bar{t} : B}\ (\text{app})$$

$$\frac{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Sq}} \vec{s} : \vec{A}}{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Tm}} x\ \vec{s} : \alpha}\ (\text{var}) \qquad \frac{\Gamma \vdash_{\mathsf{Bg}} \bar{s}_1 : A_1 \quad \cdots \quad \Gamma \vdash_{\mathsf{Bg}} \bar{s}_n : A_n}{\Gamma \vdash_{\mathsf{Sq}} \langle \bar{s}_1, \ldots, \bar{s}_n \rangle : \langle A_1, \ldots, A_n \rangle}\ (\text{seq})$$

Typing rules for the simply-typed η-long resource calculus.

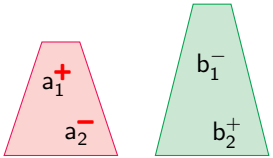
Augmentations *are* normal resource-terms

Our Goal:

$$\|\Gamma \vdash^{\mathsf{nf}}_{\mathsf{Tm}} s : A\|_{\mathsf{Tm}} \quad \in \quad \mathrm{Aug}_{\bullet}([\![\Gamma]\!] \vdash [\![A]\!])$$

$$\|\Gamma \vdash^{\mathsf{nf}}_{\mathsf{Bg}} \bar{s} : A\|_{\mathsf{Bg}} \quad \in \quad \mathrm{Aug}([\![\Gamma]\!] \vdash [\![A]\!])$$

$$\|\Gamma \vdash^{\mathsf{nf}}_{\mathsf{Sq}} \vec{s} : \vec{A}\|_{\mathsf{Sq}} \quad \in \quad \mathrm{Aug}([\![\Gamma]\!] \vdash [\![\vec{A}]\!])$$



Arena $A \vdash B$.

The Isomorphism – Resource Sequences

$$\frac{\Gamma \vdash_{\mathsf{Bg}} \bar{s}_1 : A_1 \quad \cdots \quad \Gamma \vdash_{\mathsf{Bg}} \bar{s}_n : A_n}{\Gamma \vdash_{\mathsf{Sq}} \langle \bar{s}_1, \ldots, \bar{s}_n \rangle : \langle A_1, \ldots, A_n \rangle} \ (\mathsf{seq})$$

The Isomorphism – Resource Sequences

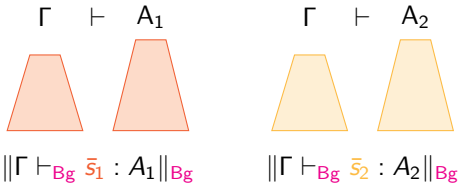
$$\frac{\Gamma \vdash_{\mathsf{Bg}} \bar{s}_1 : A_1 \quad \cdots \quad \Gamma \vdash_{\mathsf{Bg}} \bar{s}_n : A_n}{\Gamma \vdash_{\mathsf{Sq}} \langle \bar{s}_1, \ldots, \bar{s}_n \rangle : \langle A_1, \ldots, A_n \rangle} \ (\mathsf{seq})$$

Proposition. We have a bijection:

$$\langle - \rangle_{\mathsf{Aug}} \colon \prod_{i=1}^{n} \mathsf{Aug}(\Gamma \vdash A_i) \ \cong \ \mathsf{Aug}(\Gamma \vdash \bigotimes_{i=1}^{n} A_i)$$

The Isomorphism – Resource Sequences

$$\frac{\Gamma \vdash_{\mathsf{Bg}} \bar{s}_1 : A_1 \quad \cdots \quad \Gamma \vdash_{\mathsf{Bg}} \bar{s}_n : A_n}{\Gamma \vdash_{\mathsf{Sq}} \langle \bar{s}_1, \ldots, \bar{s}_n \rangle : \langle A_1, \ldots, A_n \rangle} \ (\mathsf{seq})$$

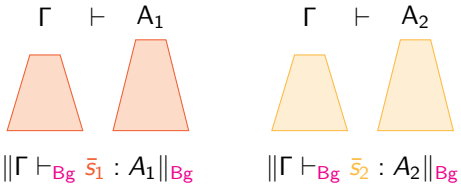


Proposition. We have a bijection:

$$\langle - \rangle_{\mathsf{Aug}} \colon \prod_{i=1}^{n} \mathsf{Aug}(\Gamma \vdash A_i) \ \cong \ \mathsf{Aug}(\Gamma \vdash \bigotimes_{i=1}^{n} A_i)$$

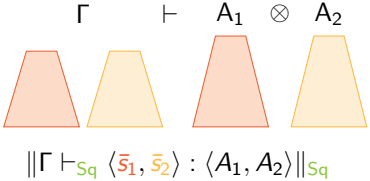
The Isomorphism – Resource Sequences

$$\frac{\Gamma \vdash_{\mathsf{Bg}} \bar{s}_1 : A_1 \quad \cdots \quad \Gamma \vdash_{\mathsf{Bg}} \bar{s}_n : A_n}{\Gamma \vdash_{\mathsf{Sq}} \langle \bar{s}_1, \ldots, \bar{s}_n \rangle : \langle A_1, \ldots, A_n \rangle}\ (\mathsf{seq})$$



Proposition. We have a bijection:

$$\langle - \rangle_{\mathsf{Aug}} \colon \prod_{i=1}^{n} \mathsf{Aug}(\Gamma \vdash A_i) \;\cong\; \mathsf{Aug}(\Gamma \vdash \bigotimes_{i=1}^{n} A_i)$$



The Isomorphism – Resource Bags

$$\frac{\Gamma \vdash_{\mathsf{Tm}} s_1 : A \quad \cdots \quad \Gamma \vdash_{\mathsf{Tm}} s_n : A}{\Gamma \vdash_{\mathsf{Bg}} [s_1, \ldots, s_n] : A}\ (\mathrm{bag})$$

The Isomorphism – Resource Bags

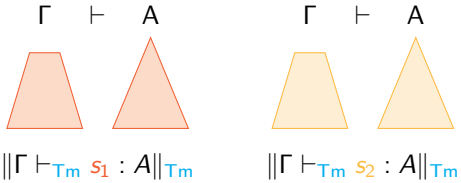
$$\frac{\Gamma \vdash_{\mathsf{Tm}} s_1 : A \quad \cdots \quad \Gamma \vdash_{\mathsf{Tm}} s_n : A}{\Gamma \vdash_{\mathsf{Bg}} [s_1, \ldots, s_n] : A}\ (\mathrm{bag})$$

Proposition. We have a bijection:

$$\Pi_{\mathsf{Aug}}(-) \colon \mathcal{M}_f(\mathsf{Aug}_\bullet(\Gamma \vdash A)) \ \cong\ \mathsf{Aug}(\Gamma \vdash A)$$

The Isomorphism – Resource Bags

$$\frac{\Gamma \vdash_{\mathsf{Tm}} s_1 : A \quad \cdots \quad \Gamma \vdash_{\mathsf{Tm}} s_n : A}{\Gamma \vdash_{\mathsf{Bg}} [s_1, \ldots, s_n] : A}\ (\mathrm{bag})$$

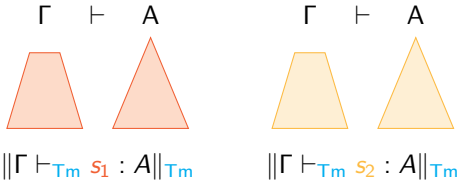


Proposition. We have a bijection:

$$\Pi_{\mathsf{Aug}}(-) \colon \mathcal{M}_f(\mathsf{Aug}_\bullet(\Gamma \vdash A)) \;\cong\; \mathsf{Aug}(\Gamma \vdash A)$$

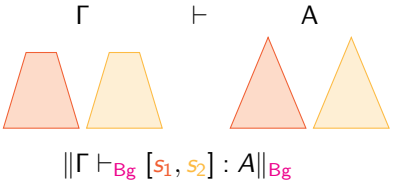
The Isomorphism – Resource Bags

$$\frac{\Gamma \vdash_{\mathsf{Tm}} s_1 : A \quad \cdots \quad \Gamma \vdash_{\mathsf{Tm}} s_n : A}{\Gamma \vdash_{\mathsf{Bg}} [s_1, \ldots, s_n] : A}\ (\text{bag})$$



Proposition. We have a bijection:

$$\Pi_{\mathsf{Aug}}(-) \colon \mathcal{M}_f(\mathsf{Aug}_\bullet(\Gamma \vdash A)) \;\cong\; \mathsf{Aug}(\Gamma \vdash A)$$



The Isomorphism – Abstractions

$$\frac{\Gamma, x : A \vdash_{\mathsf{Tm}} s : B}{\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B}\ (\mathrm{abs})$$

The Isomorphism – Abstractions

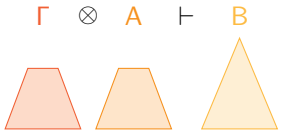
$$\frac{\Gamma, x : A \vdash_{\mathsf{Tm}} s : B}{\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B}\ (\mathrm{abs})$$

Proposition. We have a bijection:

$$\Lambda^{\mathsf{Aug}_\bullet}_{\Gamma,A,B} : \mathsf{Aug}_\bullet(\Gamma \otimes A \vdash B) \ \cong\ \mathsf{Aug}_\bullet(\Gamma \vdash A \to B)$$

The Isomorphism – Abstractions

$$\frac{\Gamma, x : A \vdash_{\mathsf{Tm}} s : B}{\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B}\ (\mathrm{abs})$$

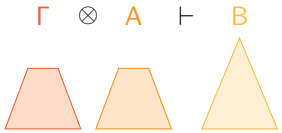


Proposition. We have a bijection:

$$\Lambda^{\mathsf{Aug}_\bullet}_{\Gamma,A,B} : \mathsf{Aug}_\bullet(\Gamma \otimes A \vdash B) \ \cong\ \mathsf{Aug}_\bullet(\Gamma \vdash A \to B)$$

The Isomorphism – Abstractions

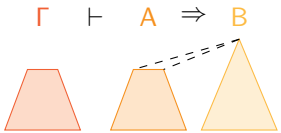
$$\frac{\Gamma, x : A \vdash_{\mathsf{Tm}} s : B}{\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B}\ (\mathrm{abs})$$



$\|\Gamma, x : A \vdash_{\mathsf{Tm}} s : B\|_{\mathsf{Tm}}$

Proposition. We have a bijection:

$$\Lambda^{\mathsf{Aug}_\bullet}_{\Gamma,A,B} : \mathsf{Aug}_\bullet(\Gamma \otimes A \vdash B) \ \cong\ \mathsf{Aug}_\bullet(\Gamma \vdash A \to B)$$



$\|\Gamma \vdash_{\mathsf{Tm}} \lambda x.s : A \to B\|_{\mathsf{Tm}}$

The Isomorphism – Applications

$$\frac{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Sq}} \vec{t} : \vec{A}}{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Tm}} x\ \vec{t} : \alpha}\ (\mathsf{var})$$

The Isomorphism – Applications

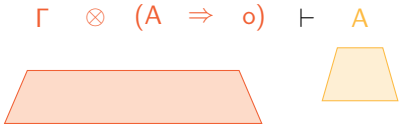
$$\frac{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Sq}} \vec{t} : \vec{A}}{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Tm}} x\ \vec{t} : \alpha}\ (\mathsf{var})$$

Proposition. We have a bijection:

$$\square(-) \colon \mathsf{Aug}(\Gamma, \vec{A} \to \alpha \vdash \vec{A}) \ \cong\ \mathsf{Aug}^{x}_{\bullet}(\Gamma, \vec{A} \to \alpha \vdash \alpha)$$

The Isomorphism – Applications

$$\frac{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Sq}} \vec{t} : \vec{A}}{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Tm}} x\ \vec{t} : \alpha}\ (\mathsf{var})$$



$$\|\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Sq}} \vec{t} : \vec{A}\|_{\mathsf{Tm}}$$

Proposition. We have a bijection:

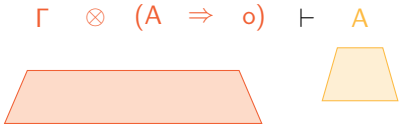
$$\Box(-) \colon \mathsf{Aug}(\Gamma, \vec{A} \to \alpha \vdash \vec{A}) \ \cong\ \mathsf{Aug}^{x}_{\bullet}(\Gamma, \vec{A} \to \alpha \vdash \alpha)$$

The Isomorphism – Applications

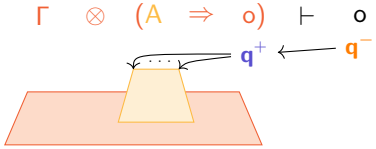
$$\frac{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Sq}} \vec{t} : \vec{A}}{\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Tm}} x\ \vec{t} : \alpha}\ (\mathsf{var})$$

Proposition. We have a bijection:

$$\square(-) \colon \mathsf{Aug}(\Gamma, \vec{A} \to \alpha \vdash \vec{A}) \ \cong\ \mathsf{Aug}^{\mathsf{x}}_{\bullet}(\Gamma, \vec{A} \to \alpha \vdash \alpha)$$



$$\|\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Sq}} \vec{t} : \vec{A}\|_{\mathsf{Tm}}$$



$$\|\Gamma, x : \vec{A} \to \alpha \vdash_{\mathsf{Tm}} x\ \vec{t} : \alpha\|_{\mathsf{Tm}}$$

Augmentation *are* normal resource-terms

Theorem. For Γ a context and A a type, there are bijections:

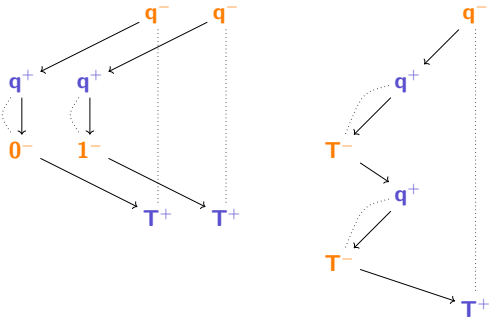
$$\begin{array}{lcll} \| - \|_{\mathsf{Tm}} & : & \mathsf{Tm}_{\mathsf{nf}}(\Gamma; A) & \cong \quad \mathsf{Aug}_{\bullet}(\llbracket \Gamma \rrbracket \vdash \llbracket A \rrbracket) \\ \| - \|_{\mathsf{Bg}} & : & \mathsf{Bg}_{\mathsf{nf}}(\Gamma; A) & \cong \quad \mathsf{Aug}(\llbracket \Gamma \rrbracket \vdash \llbracket A \rrbracket) \\ \| - \|_{\mathsf{Sq}} & : & \mathsf{Sq}_{\mathsf{nf}}(\Gamma; \vec{A}) & \cong \quad \mathsf{Aug}(\llbracket \Gamma \rrbracket \vdash \llbracket \vec{A} \rrbracket)\,. \end{array}$$

III – Dynamic isomorphism between PCG and the Resource Calculus

- Composition in PCG
- PCG as a category
- Resource categories
- PCG is a resource category

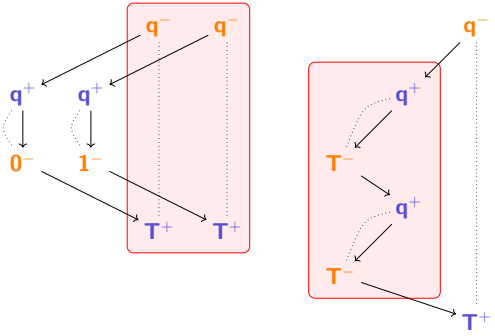
Composition

How do we compose $r : \mathbf{nat} \vdash \mathbf{bool}$ with $p : \mathbf{bool} \vdash \mathbf{bool}$?



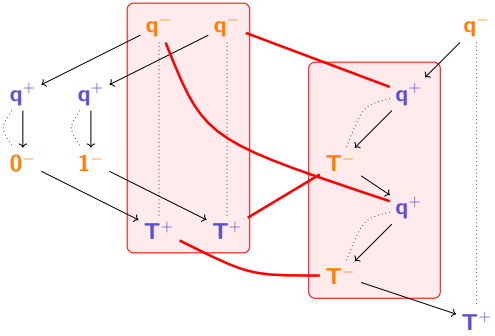
Composition

How do we compose r : **nat** $\vdash$ **bool** with p : **bool** $\vdash$ **bool** ?



Composition

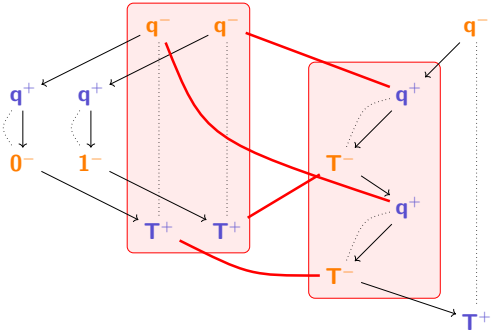
How do we compose $r : \mathbf{nat} \vdash \mathbf{bool}$ with $p : \mathbf{bool} \vdash \mathbf{bool}$?



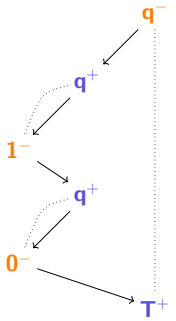
Interaction $r \circledast_{\varphi} p$

Composition

How do we compose $r : \mathbf{nat} \vdash \mathbf{bool}$ with $p : \mathbf{bool} \vdash \mathbf{bool}$?



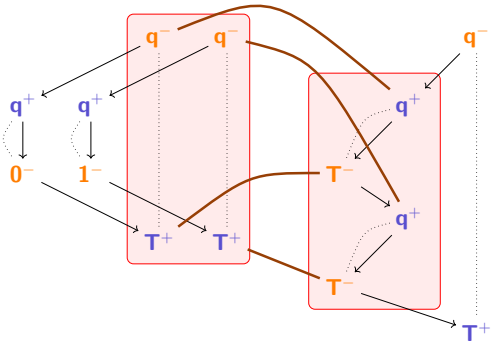
Interaction $r \circledast_{\varphi} p$



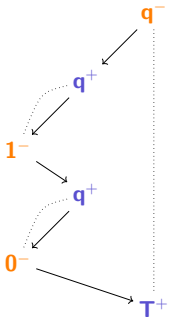
Composition $r \odot_{\varphi} p$

Composition

How do we compose $r : \mathbf{nat} \vdash \mathbf{bool}$ with $p : \mathbf{bool} \vdash \mathbf{bool}$?



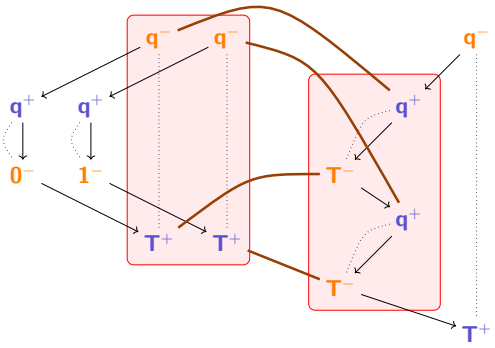
Interaction $r \circledast_{\psi} p$



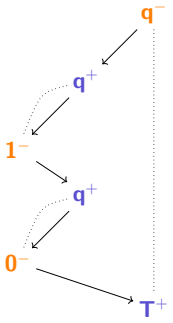
Composition $r \odot_{\varphi} p$

Composition

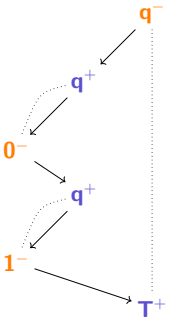
How do we compose r: **nat** $\vdash$ **bool** with p: **bool** $\vdash$ **bool** ?



Interaction $r \circledast_{\psi} p$



Composition $r \odot_{\varphi} p$



Composition $r \odot_{\psi} p$

Composition and substitution

Composition in PCG.

$$r \odot_{\varphi} p \quad \neq \quad r \odot_{\psi} p \quad !$$

Composition and substitution

Composition in PCG.

$$r \odot_{\varphi} p \quad \neq \quad r \odot_{\psi} p \quad !$$

The **composition** of r and p is:

$$p \odot r \quad \stackrel{\text{def}}{=} \quad \sum_{\varphi : r_{rhs} \cong p_{lhs}} p \odot_{\varphi} r$$

Composition and substitution

Composition in PCG.

$$r \odot_{\varphi} p \quad \neq \quad r \odot_{\psi} p \quad !$$

The **composition** of r and p is:

$$p \odot r \quad \stackrel{\text{def}}{=} \quad \sum_{\varphi : r_{rhs} \cong p_{lhs}} p \odot_{\varphi} r$$

Substitution in the resource calculus.

$$(\lambda f.f\ [x][x])\,\langle [y,z]/x \rangle \ = \lambda f.f\ [y]\ [z] \\ + \ \lambda f.f\ [z]\ [y]$$

Composition and substitution

Composition in PCG.

$$r \odot_\varphi p \quad \neq \quad r \odot_\psi p \quad !$$

The **composition** of r and p is:

$$p \odot r \quad \stackrel{\text{def}}{=} \quad \sum_{\varphi : r_{rhs} \cong p_{lhs}} p \odot_\varphi r$$

Substitution in the resource calculus.

$$(\lambda f.f\ [x][x])\,\langle [y,z]/x \rangle \ = \lambda f.f\ [y]\ [z] \\ + \ \lambda f.f\ [z]\ [y]$$

Intuitively, the **substitution** $s\langle \bar{t}/x \rangle$ is:

$$s\langle \bar{t}/x \rangle \ = \ \text{“} \sum_{\varphi :\ \substack{\text{occurrences} \\ \text{of } x \text{ in } s}} s\langle \bar{t}/x \rangle_\varphi \ \text{”} \quad \cong \bar{t}$$

Strategies in PCG

Strategies in PCG are sums of augmentations:

$$\sigma = \sum_{p \in \mathsf{Aug}(A)} \sigma(p) \cdot p \,.$$

Strategies in PCG

Strategies in PCG are sums of augmentations:

$$\sigma = \sum_{p \in \mathsf{Aug}(\mathsf{A})} \sigma(p) \cdot p \,.$$

The **composition** of the strategies $\sigma : \mathsf{A} \vdash \mathsf{B}$ and $\tau : \mathsf{B} \vdash \mathsf{C}$ is:

$$\tau \odot \sigma \stackrel{\text{def}}{=} \sum_{\substack{p \in \mathsf{Aug}(\mathsf{A} \vdash \mathsf{B}) \\ r \in \mathsf{Aug}(\mathsf{B} \vdash \mathsf{C})}} \sigma(p)\tau(r) \cdot (r \odot p) \,.$$

The *copycat* strategy

Consider $x \in \mathsf{Conf}(\mathsf{A})$.
The **copycat augmentation** on x is:

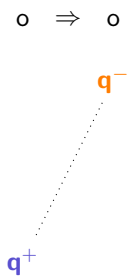
$$cc_x = \text{“}x \vdash x\text{”} \ .$$

The *copycat* strategy

Consider $x \in \mathrm{Conf}(\mathrm{A})$.
The **copycat augmentation** on x is:

$$cc_x = \text{“}x \vdash x\text{”} .$$

Example.



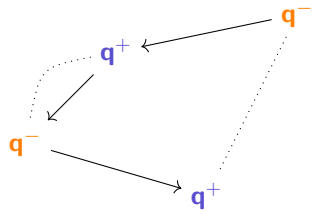
The *copycat* strategy

Consider $x \in \mathrm{Conf(A)}$.
The **copycat augmentation** on x is:

$$cc_x = \text{“}x \vdash x\text{”} .$$

Example.

$$\mathrm{o} \Rightarrow \mathrm{o} \vdash \mathrm{o} \Rightarrow \mathrm{o}$$



The *copycat* strategy

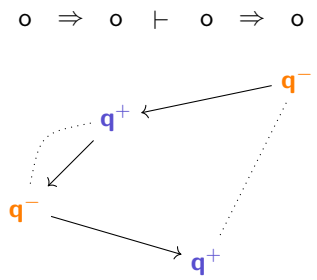
Consider $x \in \mathsf{Conf(A)}$.
The **copycat augmentation** on x is:

$$cc_x = \text{“}x \vdash x\text{”} \,.$$

The **copycat strategy** on A is:

$$\mathsf{id_A} \stackrel{\text{def}}{=} \sum_{x \in \mathsf{Conf(A)}} \frac{1}{S(x)} \cdot cc_x \,.$$

Example.



PCG as a category

PCG is a category with:

- **Arenas** as objects;
- **Strategies** as morphisms;
- $\odot$ as the composition;
- **Copycats** as identities.

PCG as a category

PCG is a category with:

- **Arenas** as objects;
- **Strategies** as morphisms;
- $\odot$ as the composition;
- **Copycats** as identities.

Theorem. PCG is a SMCC.

PCG as a category

PCG is a category with:

- **Arenas** as objects;
- **Strategies** as morphisms;
- $\odot$ as the composition;
- **Copycats** as identities.

Theorem. PCG is a SMCC.

Theorem. There is a structure preserving functor from PCG to HO.

Resource Categories

A **resource category** is an additive symmetric monoidal category where **each** object A has a bialgebra structure and **a pointed identity** $\mathrm{id}_A^\bullet$.

Resource Categories

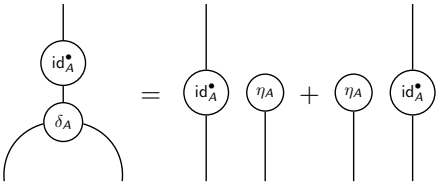
A **resource category** is an additive symmetric monoidal category where **each** object A has a bialgebra structure and **a pointed identity** $\mathrm{id}^\bullet_A$.

A **pointed identity** $\mathrm{id}^\bullet_A$ is:

- *idempotent:* $\mathrm{id}^\bullet_A \circ \mathrm{id}^\bullet_A = \mathrm{id}^\bullet_A$
- *non-erasable:* $\varepsilon_A \circ \mathrm{id}^\bullet_A = 0$
- *non-erasing:* $\mathrm{id}^\bullet_A \circ \eta_A = 0$

Resource Categories

A **resource category** is an additive symmetric monoidal category where **each** object A has a bialgebra structure and **a pointed identity** $\mathrm{id}_A^\bullet$.



A **pointed identity** $\mathrm{id}_A^\bullet$ is:

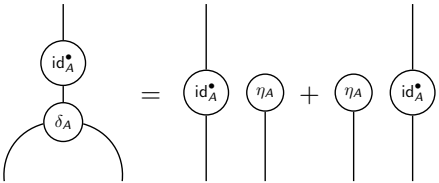
- *idempotent:* $\mathrm{id}_A^\bullet \circ \mathrm{id}_A^\bullet = \mathrm{id}_A^\bullet$
- *non-erasable:* $\varepsilon_A \circ \mathrm{id}_A^\bullet = 0$
- *non-erasing:* $\mathrm{id}_A^\bullet \circ \eta_A = 0$
- *non-duplicable*

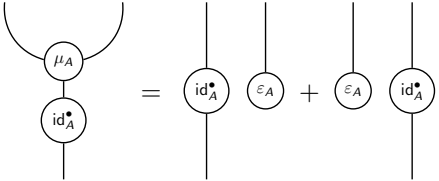
Resource Categories

A **resource category** is an additive symmetric monoidal category where **each** object A has a bialgebra structure and **a pointed identity** $\mathrm{id}^\bullet_A$.

A **pointed identity** $\mathrm{id}^\bullet_A$ is:

- *idempotent:* $\mathrm{id}^\bullet_A \circ \mathrm{id}^\bullet_A = \mathrm{id}^\bullet_A$
- *non-erasable:* $\varepsilon_A \circ \mathrm{id}^\bullet_A = 0$
- *non-erasing:* $\mathrm{id}^\bullet_A \circ \eta_A = 0$
- *non-duplicable*
- *non-duplicating*





Pointed morphisms

A morphism $f \in \mathcal{C}(A, B)$ is **pointed**, noted $f \in \mathcal{C}_\bullet(A, B)$, if:

$$\mathrm{id}_B^\bullet \circ f = f$$

Pointed morphisms

A morphism $f \in \mathcal{C}(A, B)$ is **pointed**, noted $f \in \mathcal{C}_\bullet(A, B)$, if:

$$\mathrm{id}^\bullet_B \circ f = f$$

Intuition:

pointed morphisms
$=$ resource terms
$=$ pointed augmentations

Pointed morphisms

A morphism $f \in \mathcal{C}(A, B)$ is **pointed**, noted $f \in \mathcal{C}_\bullet(A, B)$, if:

$$\mathrm{id}^\bullet_B \circ f = f$$

Intuition:

pointed morphisms
$=$ resource terms
$=$ pointed augmentations

Our Goal:

$$[\![\Gamma \vdash_{\mathsf{Tm}} s : A]\!] \in \mathcal{C}_\bullet([\![\Gamma]\!] \vdash [\![A]\!])$$

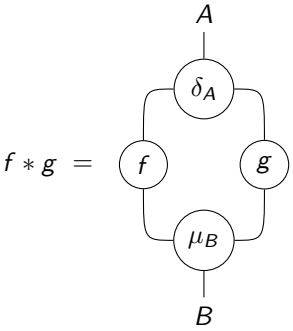
$$[\![\Gamma \vdash_{\mathsf{Bg}} \bar{s} : A]\!] \in \mathcal{M}_f(\mathcal{C}_\bullet([\![\Gamma]\!] \vdash [\![A]\!]))$$

$$[\![\Gamma \vdash_{\mathsf{Sq}} \vec{s} : \vec{A}]\!] \in \left\langle \mathcal{M}_f(\mathcal{C}_\bullet([\![\Gamma]\!] \vdash [\![A_i]\!])) \right\rangle_{1 \le i \le n}$$

Dealing with multisets of pointed morphisms

Consider $f : A \to B$ and $g : A \to B$.
Their **union** $f * g : A \to B$ is:

$$f * g \stackrel{\text{def}}{=} \mu_B \circ (f \otimes g) \circ \delta_A$$

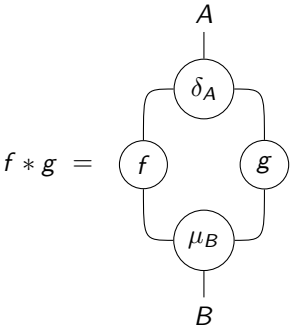


Dealing with multisets of pointed morphisms

Consider $f : A \to B$ and $g : A \to B$.
Their **union** $f * g : A \to B$ is:

$$f * g \stackrel{\text{def}}{=} \mu_B \circ (f \otimes g) \circ \delta_A$$

It's similar to $\Pi_{\text{Aug}}[-]$*!*



“Linear” behaviour of pointed morphisms

Key Lemma.
Consider $\bar{f} \in \mathcal{M}_f(\mathcal{C}_\bullet(A, B))$, then:

1. “Non erasable”:

$$\varepsilon_B \circ \prod \bar{f} = \begin{cases} 1 & \text{if } \bar{f} = [\,] \\ 0 & \text{otherwise} \end{cases},$$

“Linear” behaviour of pointed morphisms

Key Lemma.
Consider $\bar{f} \in \mathcal{M}_f(\mathcal{C}_\bullet(A, B))$, then:

1. “Non erasable”:

$$\varepsilon_B \circ \prod \bar{f} = \begin{cases} 1 & \text{if } \bar{f} = [\,] \\ 0 & \text{otherwise} \end{cases},$$

2. “Non duplicable”.

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & A \otimes A \\ \Big\downarrow{\scriptstyle \prod \bar{f}} & & \Big\downarrow{\scriptstyle \sum_{\bar{f} \lhd \bar{f}_1 * \bar{f}_2} \prod \bar{f}_1 \otimes \prod \bar{f}_2} \\ B & \xrightarrow[\delta_B]{} & B \otimes B \end{array}$$

A sound interpretation

Theorem (Interpretation). We have an interpretation:

$$[\![\Gamma \vdash_{\mathsf{Tm}} s : A]\!] \quad \in \quad \mathcal{C}_\bullet([\![\Gamma]\!] \vdash [\![A]\!])$$

$$[\![\Gamma \vdash_{\mathsf{Bg}} \bar{s} : A]\!] \quad \in \quad \mathcal{M}_f(\mathcal{C}_\bullet([\![\Gamma]\!] \vdash [\![A]\!]))$$

$$[\![\Gamma \vdash_{\mathsf{Sq}} \vec{s} : \vec{A}]\!] \quad \in \quad \left\langle \mathcal{M}_f(\mathcal{C}_\bullet([\![\Gamma]\!] \vdash [\![A_i]\!])) \right\rangle_{1 \leq i \leq n}$$

A sound interpretation

Theorem (Interpretation). We have an interpretation:

$$\llbracket \Gamma \vdash_{\mathsf{Tm}} s : A \rrbracket \quad \in \quad \mathcal{C}_\bullet(\llbracket \Gamma \rrbracket \vdash \llbracket A \rrbracket)$$

$$\llbracket \Gamma \vdash_{\mathsf{Bg}} \bar{s} : A \rrbracket \quad \in \quad \mathcal{M}_f(\mathcal{C}_\bullet(\llbracket \Gamma \rrbracket \vdash \llbracket A \rrbracket))$$

$$\llbracket \Gamma \vdash_{\mathsf{Sq}} \vec{s} : \vec{A} \rrbracket \quad \in \quad \left\langle \mathcal{M}_f(\mathcal{C}_\bullet(\llbracket \Gamma \rrbracket \vdash \llbracket A_i \rrbracket)) \right\rangle_{1 \leq i \leq n}$$

Theorem (Soundness). Consider $S \in \Sigma\mathsf{Tm}(\Gamma; A)$.
If $S \rightsquigarrow S'$, then $\llbracket S \rrbracket = \llbracket S' \rrbracket$.

Pointed morphisms in PCG

Pointed identities in PCG are:

$$\mathrm{id}_A^\bullet \overset{\mathrm{def}}{=} \sum_{x \in \mathrm{Conf}_\bullet(A)} \frac{1}{S(x)} \cdot \mathfrak{a}_x \,.$$

Pointed morphisms in PCG

Pointed identities in PCG are:

$$\mathrm{id}_A^\bullet \overset{\mathrm{def}}{=} \sum_{x \in \mathrm{Conf}_\bullet(A)} \frac{1}{S(x)} \cdot \alpha_x \,.$$

Lemma. Pointed morphisms in PCG are:

$$\sigma = \sum_{p \in \mathrm{Aug}_\bullet(A)} \sigma(p) \cdot p \,.$$

Pointed morphisms in PCG

Pointed identities in PCG are:

$$\mathrm{id}_A^\bullet \stackrel{\mathrm{def}}{=} \sum_{x \in \mathrm{Conf}_\bullet(A)} \frac{1}{S(x)} \cdot \mathrm{ax}_x \,.$$

Lemma. Pointed morphisms in PCG are:

$$\sigma = \sum_{p \in \mathrm{Aug}_\bullet(A)} \sigma(p) \cdot p \,.$$

Theorem. PCG is a resource category.

PCG is a Resource Category

Theorem. PCG is a resource category.

PCG is a Resource Category

Theorem. PCG is a resource category.

Corollary. There is a sound interpretation $[\![-]\!] : \Delta \to \mathrm{PCG}$.

PCG is a Resource Category

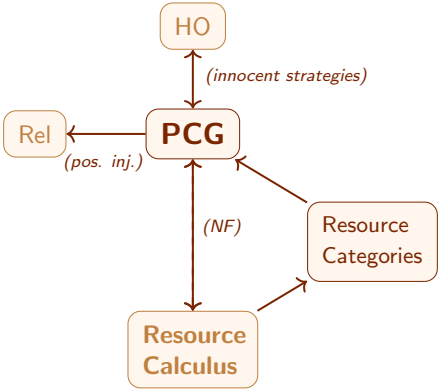
Theorem. PCG is a resource category.

Corollary. There is a sound interpretation $[\![-]\!] : \Delta \to \mathsf{PCG}$.

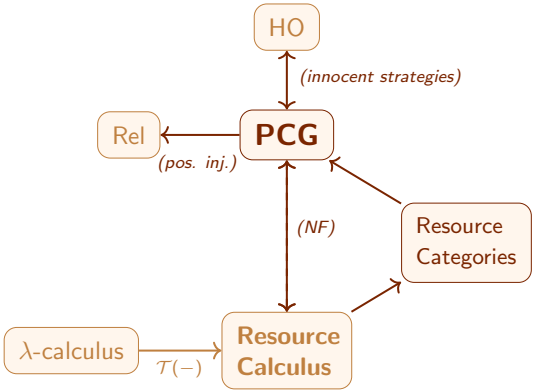
Theorem. Consider $s \in \mathsf{Tm}_{\mathsf{nf}}(\Gamma; A)$. Then $[\![s]\!] = 1 \cdot \|s\|_{\mathsf{Tm}}$.

Normalising a resource term is computing its semantics!

Conclusion and Perspectives



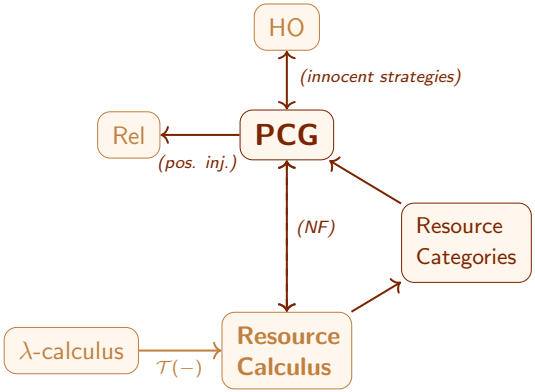
Conclusion and Perspectives



Perspectives:

1. Taylor expansion?

Conclusion and Perspectives

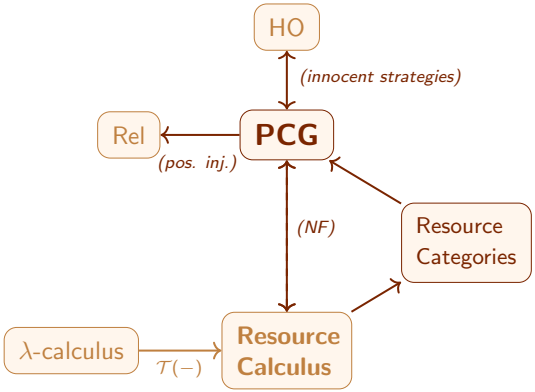


Perspectives:

1. Taylor expansion?

$$\begin{array}{ccccc} M & \longrightarrow & \mathcal{T}(M) & \longrightarrow & \mathsf{NF}(\mathcal{T}(M)) \\ \downarrow & & \downarrow & & \downarrow \\ [\![M]\!] & = & [\![\mathcal{T}(M)]\!] & = & \|\mathsf{NF}(\mathcal{T}(M))\| \end{array}$$

Conclusion and Perspectives

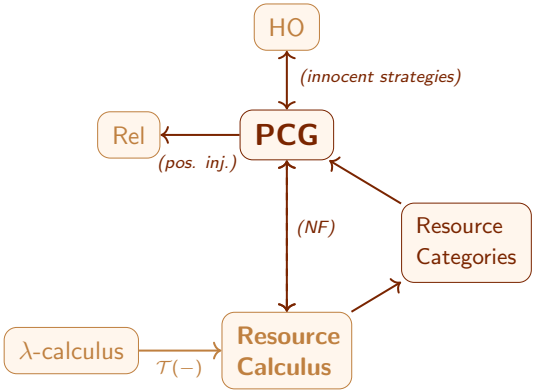


Perspectives:

1. Taylor expansion?

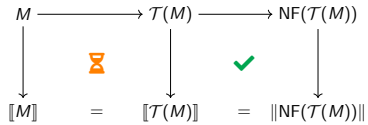
$$\begin{array}{ccccc} M & \longrightarrow & \mathcal{T}(M) & \longrightarrow & \mathsf{NF}(\mathcal{T}(M)) \\ \downarrow & & \downarrow & & \downarrow \\ [\![M]\!] & = & [\![\mathcal{T}(M)]\!] & = & \|\mathsf{NF}(\mathcal{T}(M))\| \end{array}$$

Conclusion and Perspectives



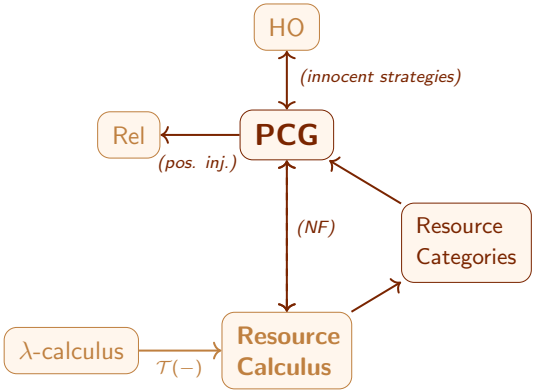
Perspectives:

1. Taylor expansion?



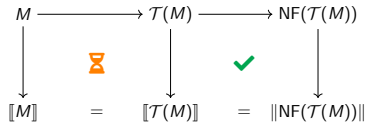
2. Untyped calculi?

Conclusion and Perspectives



Perspectives:

1. Taylor expansion?



2. Untyped calculi?

Thanks!!