

Game Semantics of Taylor Expansion

Lison Blondeau-Patissier (LIP, ENS de Lyon)

Joint work with Pierre Clairambault (LIS, Aix-Marseille Université)
and Lionel Vaux Auclair (I2M, Aix-Marseille Université)

Trends in Linear Logic and Applications, 19 July 2026

*Note: This talk was prepared during some of the worst heatwaves in my life.
I did my best but the talk certainly suffered from it.*

- I – Towards an Extensional λ -theory
- II – Extensional Resource Calculus
- III – Game Semantics: Pointer Concurrent Games

The λ -calculus

Terms of the usual λ -calculus:

$$M, N ::= x \mid \lambda x.M \mid M N$$

The λ -calculus

Terms of the usual λ -calculus:

$$M, N ::= x \mid \lambda x.M \mid M N$$

1. α -equivalence: $M =_{\alpha} M \{y/x\}$ for y fresh
2. β -congruence: $(\lambda x.M)N =_{\beta} M \{N/x\}$
3. η -congruence: $M =_{\eta} \lambda x.M x$

The λ -calculus

Terms of the usual λ -calculus:

$$M, N ::= x \mid \lambda x.M \mid M N$$

1. α -equivalence: $M =_\alpha M \{y/x\}$ for y fresh
2. β -congruence: $(\lambda x.M)N =_\beta M \{N/x\}$
3. η -congruence: $M =_\eta \lambda x.M x$

A λ -**theory** is a congruence relation containing **1 + 2**.

It is **extensional** if it also contains **3**.

The λ -calculus

Terms of the usual λ -calculus:

$$M, N ::= x \mid \lambda x.M \mid M N$$

1. α -equivalence: $M =_\alpha M \{y/x\}$ for y fresh
2. β -congruence: $(\lambda x.M)N =_\beta M \{N/x\}$
3. η -congruence: $M =_\eta \lambda x.M x$

A λ -**theory** is a congruence relation containing **1 + 2**.

It is **extensional** if it also contains **3**.

A term M is **solvable** if

$$M =_\beta \lambda x_1. \dots \lambda x_n. x_i M_1 \dots M_k$$

and **unsolvable** otherwise.

A λ -theory is **sensible** if:

- it never equates a solvable term and an unsolvable term;
- it equates all unsolvable terms.

The Resource Calculus

Terms of the resource calculus:

$$m, n ::= x \mid \lambda x. m \mid m \bar{n} \qquad \bar{m} ::= [m_1, \dots, m_k]$$

Example:

$$M := \lambda x. \lambda y. x \ y$$

$$m := \lambda x. \lambda y. x \ [y, y]$$

The Resource Calculus

Terms of the resource calculus:

$$m, n ::= x \mid \lambda x. m \mid m \bar{n} \qquad \bar{m} ::= [m_1, \dots, m_k]$$

Example: $M := \lambda x. \lambda y. x \ y$ $m := \lambda x. \lambda y. x \ [y, y]$

Substitution and β -reduction:

$$m[\bar{n}/x] := \begin{cases} \sum_{\sigma \in \mathcal{S}(k)} m \{n_{\sigma(1)}, \dots, n_{\sigma(k)} / x_1, \dots, x_k\} & \text{if } |\bar{n}| = \sharp \text{ occurrences of } x \\ 0 & \text{otherwise} \end{cases}$$

The β -**reduction** is defined with $(\lambda x. m) \bar{n} \rightarrow_{\beta} m[\bar{n}/x]$.

Example: $(\lambda x. \lambda y. x \ [y, x]) \ [\lambda z. z, \lambda z. z] \rightarrow_{\beta} 2 \cdot \lambda y. (\lambda z. z) \ [y, \lambda z. z]$

Taylor Expansion

Taylor expansion:

$$\mathcal{T}(x) := x \quad \mathcal{T}(\lambda x.M) := \lambda x.\mathcal{T}(M) \quad \mathcal{T}(M \ N) := \sum_{k \in \mathbb{N}} \frac{1}{k!} \mathcal{T}(M) \ \mathcal{T}(N)^k$$

Example:

$$\mathcal{T}(\lambda x.y \ x) = 1 \cdot \lambda x.y \ [] + 1 \cdot \lambda x.y \ [x] + \frac{1}{2} \cdot \lambda x.y \ [x, x] + \dots$$

The λ -theory $=_{\mathcal{T}}$

We set the relation $=_{\mathcal{T}}$ with:

$$M =_{\mathcal{T}} N \quad \text{iff} \quad \text{NF}(\mathcal{T}(M)) = \text{NF}(\mathcal{T}(N))$$

Example:

$$\begin{aligned} \mathcal{T}((\lambda x.x) y) &= (\lambda x.x) [] + (\lambda x.x) [y] + \frac{1}{2} \cdot (\lambda x.x) [y, y] + \dots \\ &\rightarrow_{\beta}^* y \end{aligned}$$

Hence $(\lambda x.x) y =_{\mathcal{T}} y$.

The λ -theory $=_{\mathcal{T}}$

We set the relation $=_{\mathcal{T}}$ with:

$$M =_{\mathcal{T}} N \quad \text{iff} \quad \text{NF}(\mathcal{T}(M)) = \text{NF}(\mathcal{T}(N))$$

Example:

$$\begin{aligned} \mathcal{T}((\lambda x.x) y) &= (\lambda x.x) [] + (\lambda x.x) [y] + \frac{1}{2} \cdot (\lambda x.x) [y, y] + \dots \\ &\rightarrow_{\beta}^* y \end{aligned}$$

Hence $(\lambda x.x) y =_{\mathcal{T}} y$.

Proposition 1.

The relation $=_{\mathcal{T}}$ is a λ -theory.

Proposition 2.

The λ -theory $=_{\mathcal{T}}$ is **sensible**.

What about extensionnality?

Reminder: A λ -theory is **extensional** if it contains the η -rule.

$$\begin{array}{lcl} \mathcal{T}(y) & = & \\ \mathcal{T}(\lambda x. y \ x) & = & \end{array}$$

What about extensionnality?

Reminder: A λ -theory is **extensional** if it contains the η -rule.

$$\begin{array}{lcl} \mathcal{T}(y) & = & y \\ \mathcal{T}(\lambda x. y \ x) & = & \end{array}$$

What about extensionnality?

Reminder: A λ -theory is **extensional** if it contains the η -rule.

$$\begin{aligned}\mathcal{T}(y) &= y \\ \mathcal{T}(\lambda x.y \ x) &= \lambda x.y \ [] + \lambda x.y \ [x] + \dots\end{aligned}$$

What about extensionnality?

Reminder: A λ -theory is **extensional** if it contains the η -rule.

$$\begin{aligned}\mathcal{T}(y) &= y \\ \mathcal{T}(\lambda x.y \ x) &= \lambda x.y \ [] + \lambda x.y \ [x] + \dots\end{aligned}$$

Can we construct $\mathcal{T}_\eta$ such that $=_{\mathcal{T}_\eta}$ is extensional?

Extensionality for typed calculi (1/2)

Simple types: $A, B ::= \alpha \mid A \rightarrow B$

Notation: $\langle A_1, \dots, A_n \rangle \rightarrow \alpha := (A_1 \rightarrow (\dots A_n \rightarrow \alpha) \dots)$

Extensionality for typed calculi (1/2)

Simple types: $A, B ::= \alpha \mid A \rightarrow B$

Notation: $\langle A_1, \dots, A_n \rangle \rightarrow \alpha := (A_1 \rightarrow (\dots A_n \rightarrow \alpha) \dots)$

Solution 1: Expand the λ -terms themselves.

For any term of type $\langle A_1, \dots, A_n \rangle \rightarrow \alpha$, we set:

$$\begin{aligned}\eta(y) &:= \lambda x_1. \dots \lambda x_n. y \ \eta(x_1) \ \dots \ \eta(x_n) \\ \eta(\lambda x. M) &:= \lambda x. \eta(M) \\ \eta(M \ N) &:= \lambda x_1. \dots \lambda x_n. \eta(M) \ \eta(N) \ \eta(x_1) \ \dots \ \eta(x_n)\end{aligned}$$

Then we just have to set $\mathcal{T}_\eta := \mathcal{T} \circ \eta$.

Example:

$$\begin{array}{ll} y & : \quad \alpha \rightarrow \alpha \\ \lambda x. y \ x & : \quad \alpha \rightarrow \alpha \end{array} \qquad \begin{array}{ll} \eta(y) & = \quad \lambda x. y \ x \\ \eta(\lambda x. y \ x) & = \quad \lambda x. y \ x \end{array}$$

Extensionality for typed calculi (2/2)

Solution 2: Expand the *resource* terms.

For any term of type $\langle A_1, \dots, A_n \rangle \rightarrow \alpha$, we set:

$$\begin{aligned}\mathcal{T}_\eta(y) &:= \lambda x_1 \dots \lambda x_n. y \, \mathcal{T}_\eta(x_1)^! \dots \mathcal{T}_\eta(x_n)^! \\ \mathcal{T}_\eta(\lambda x. M) &:= \lambda x. \mathcal{T}_\eta(M) \\ \mathcal{T}_\eta(M \, N) &:= \lambda x_1 \dots \lambda x_n. \mathcal{T}_\eta(M) \, \mathcal{T}_\eta(N)^! \, \mathcal{T}_\eta(x_1)^! \dots \mathcal{T}_\eta(x_n)^!\end{aligned}$$

where $\mathcal{T}_\eta(M)^!$ is the **promotion**:

$$\mathcal{T}_\eta(M)^! := \sum_{k \in \mathbb{N}} \frac{1}{k!} \mathcal{T}_\eta(M)^k$$

But what about untyped calculi?

Idea 1: Let us try to expand the λ -terms themselves...

Problem: we can have infinite η -expansion!

$$y =_{\eta} \lambda x_1. y \ x_1 =_{\eta} \lambda x_1. \lambda x_2. y \ x_1 \ x_2 =_{\eta} \dots =_{\eta} \lambda x_1. \dots \lambda x_{42}. y \ x_1 \ \dots \ x_{42} =_{\eta} \dots$$

“ η -long” forms: we want something like “ $\eta(y) = \lambda \vec{x}. y \ \eta(\vec{x})$ ”.

But what about untyped calculi?

Idea 1: Let us try to expand the λ -terms themselves...

Problem: we can have infinite η -expansion!

$$y =_{\eta} \lambda x_1. y \ x_1 =_{\eta} \lambda x_1. \lambda x_2. y \ x_1 \ x_2 =_{\eta} \dots =_{\eta} \lambda x_1. \dots \lambda x_{42}. y \ x_1 \ \dots \ x_{42} =_{\eta} \dots$$

“ η -long” forms: we want something like “ $\eta(y) = \lambda \vec{x}. y \ \eta(\vec{x})$ ”.

Idea 2: We expand the resource terms.

$\Rightarrow$ Extensional resource calculus.

- I – Towards an Extensional λ -theory
- II – Extensional Resource Calculus
- III – Game Semantics: Pointer Concurrent Games

Notations: bags and streams

Bags of any set X are multisets over X , denoted by $\bar{a}, \bar{b}, \bar{c}, \dots$

Notations: We write $[]$ for the empty bag and $\bar{a} * \bar{b}$ for the concatenation.

Streams of any set X are tuples of bags of X , written $\vec{a} := \langle \bar{a}_i \rangle_{i \in \mathbb{N}}$, **such that only a finite number of $\bar{a}_i$'s are non empty**.

Notations:

- The empty stream is denoted $\iota := \langle [] \rangle_{i \in \mathbb{N}}$
- The cons operation is $(\bar{a} :: \vec{b})_i := \begin{cases} \bar{a} & \text{if } i = 0 \\ \bar{b}_j & \text{if } i = j + 1 \end{cases}$ (writing $\vec{b} = \langle \bar{b}_j \rangle_{j \in \mathbb{N}}$)

Syntax – Variables

We will need two notions of variables :

Value variables: $x, y, z \in \mathcal{V}$

Stream variables: $\vec{x}, \vec{y}, \vec{z} \in \mathcal{V}_S$

- A stream variable $\vec{x} \in \mathcal{V}_S$ is a map $\vec{x}: i \in \mathbb{N} \mapsto \vec{x}(i) \in \mathcal{V}$.
- Stream variables partition value variables.

Syntax – Terms

Value terms $m, n, p \in \Delta_v$

Base terms $a, b, c \in \Delta_b$

Bag terms $\bar{m}, \bar{n}, \bar{p} \in \Delta_!$

Stream terms $\vec{m}, \vec{n}, \vec{p} \in \Delta_s$

Syntax – Terms

Value terms $m, n, p \in \Delta_v$

Base terms $a, b, c \in \Delta_b$

Bag terms $\bar{m}, \bar{n}, \bar{p} \in \Delta_!$

Stream terms $\vec{m}, \vec{n}, \vec{p} \in \Delta_s$

$$\begin{array}{c}
 \frac{\vec{x} \in \mathcal{V}_S \quad a \in \Delta_b}{\lambda \vec{x}. a \in \Delta_v} (\lambda) \quad \frac{m \in \Delta_v \quad \vec{n} \in \Delta_s}{m \vec{n} \in \Delta_b} (\text{app}) \quad \frac{x \in \mathcal{V} \quad \vec{n} \in \Delta_s}{x \vec{n} \in \Delta_b} (\mathcal{V}) \\
 \\
 \frac{m_1 \in \Delta_v \quad \dots \quad m_k \in \Delta_v}{[m_1, \dots, m_k] \in \Delta_!} (!) \quad \frac{}{\iota \in \Delta_s} (\iota) \quad \frac{\bar{m} \in \Delta_! \quad \vec{n} \in \Delta_s}{\bar{m} :: \vec{n} \in \Delta_s} (::)
 \end{array}$$

Extensional Resource Terms

Syntax – Terms

Example:

$$\begin{array}{c}
 \vec{x} \in \mathcal{V}_S \quad \frac{\vec{y} \in \mathcal{V}_S \quad \frac{x \in \mathcal{V} \quad \frac{}{\iota \in \Delta_S} (\iota) (\mathcal{V})}{x \iota \in \Delta_b} (\lambda)}{\lambda \vec{y}. x \iota \in \Delta_v} (\lambda) \quad \frac{\vec{y} \in \mathcal{V}_S \quad \frac{z \in \mathcal{V} \quad \frac{}{\iota \in \Delta_S} (\iota) (\mathcal{V})}{z \iota \in \Delta_b} (\lambda)}{\lambda \vec{y}. z \iota \in \Delta_v} (!) \quad \frac{}{\iota \in \Delta_S} (\iota) (\mathcal{V})}{[\lambda \vec{y}. z \iota] \in \Delta_!} (!) \quad \frac{[\lambda \vec{y}. z \iota] :: \iota \in \Delta_S}{[\lambda \vec{y}. z \iota] :: \iota \in \Delta_S} (:) \\
 \frac{\vec{x} \in \mathcal{V}_S \quad \frac{(\lambda \vec{y}. x \iota) ([\lambda \vec{y}. z \iota] :: \iota) \in \Delta_b}{(\lambda \vec{y}. x \iota) ([\lambda \vec{y}. z \iota] :: \iota) \in \Delta_b} (\lambda)}{\lambda \vec{x}. \left((\lambda \vec{y}. x \iota) ([\lambda \vec{y}. z \iota] :: \iota) \right) \in \Delta_v} (\text{app})
 \end{array}$$

Resource Substitution

The **resource substitution** for $q \in \mathcal{V}$, Δ_v , Δ_b , $\Delta_!$ or Δ_S ; $x \in \mathcal{V}$; $\bar{n} \in \Delta_!$ is:

$$q[\bar{n}/x] := \begin{cases} \sum_{\sigma \in \mathcal{S}_k} q \{n_1, \dots, n_k / x_{\sigma(1)}, \dots, x_{\sigma(k)}\} & \text{if } |\bar{n}| = \# \text{ occurrences of } x \\ 0 & \text{otherwise} \end{cases}$$

Resource Substitution

The **resource substitution** for $q \in \mathcal{V}$, Δ_v , Δ_b , $\Delta_!$ or Δ_S ; $x \in \mathcal{V}$; $\bar{n} \in \Delta_!$ is:

$$q[\bar{n}/x] := \begin{cases} \sum_{\sigma \in \mathcal{S}_k} q \{n_1, \dots, n_k / x_{\sigma(1)}, \dots, x_{\sigma(k)}\} & \text{if } |\bar{n}| = \# \text{ occurrences of } x \\ 0 & \text{otherwise} \end{cases}$$

Examples:

$$\left(\lambda \vec{y}. x \left([\lambda \vec{z}. x \iota] :: \iota \right) \right) [[m]/x] = 0$$

$$\begin{aligned} \left(\lambda \vec{y}. x \left([\lambda \vec{z}. x \iota] :: \iota \right) \right) [[m, n]/x] &= \lambda \vec{y}. m \left([\lambda \vec{z}. n \iota] :: \iota \right) \\ &+ \lambda \vec{y}. n \left([\lambda \vec{z}. m \iota] :: \iota \right) \end{aligned}$$

$$\left(\lambda \vec{y}. x \left([\lambda \vec{z}. x \iota] :: \iota \right) \right) [[m, n, p]/x] = 0$$

Shifting and erasing

We define the **shifting operations** $q[\vec{x} \uparrow]$ and $q[\vec{x} \downarrow]$:

$$\left(\lambda \vec{y}. \vec{x}(1) ([\lambda \vec{z}. \vec{x}(3) \iota] :: \iota) \right) [\vec{x} \uparrow] = \lambda \vec{y}. \vec{x}(2) ([\lambda \vec{z}. \vec{x}(4) \iota] :: \iota)$$

$$\left(\lambda \vec{y}. \vec{x}(1) ([\lambda \vec{z}. \vec{x}(3) \iota] :: \iota) \right) [\vec{x} \downarrow] = \lambda \vec{y}. \vec{x}(0) ([\lambda \vec{z}. \vec{x}(2) \iota] :: \iota)$$

We define the **erasure** $q \not\downarrow \vec{x}$ as:

$$q \not\downarrow \vec{x} := \begin{cases} q & \text{if no } \vec{x}(i) \text{ occurs in } q \\ 0 & \text{otherwise} \end{cases}$$

Fine-step reduction

The **fine-step reduction** is given by the basic rules:

$$\left(\lambda \vec{x}. a \right) \left(\bar{n} :: \vec{p} \right) \quad \mapsto_r \quad \left(\lambda \vec{x}. a \left[\bar{n} / \vec{x}(0) \right] \left[\vec{x} \downarrow \right] \right) \vec{p}$$

extended to be applied under any context.

Example:

$$\begin{aligned} & \left(\lambda \vec{x}. \vec{x}(1) \left(\left[\lambda \vec{y}. \vec{x}(1) \iota \right] :: \iota \right) \right) \left([] :: [\textcolor{blue}{m}, \textcolor{blue}{m}] :: \iota \right) \\ & \mapsto_r \quad \left(\lambda \vec{x}. \textcolor{red}{\vec{x}}(0) \left(\left[\lambda \vec{y}. \textcolor{red}{\vec{x}}(0) \iota \right] :: \iota \right) \right) \left([\textcolor{blue}{m}, \textcolor{blue}{m}] :: \iota \right) \\ & \mapsto_r \quad 2 \cdot \left(\lambda \vec{x}. \textcolor{blue}{m} \left(\left[\lambda \vec{y}. \textcolor{blue}{m} \iota \right] :: \iota \right) \right) \iota \end{aligned}$$

Fine-step reduction

The **fine-step reduction** is given by the basic rules:

$$\left(\lambda \vec{x}. a \right) \left(\bar{n} :: \vec{p} \right) \quad \mapsto_r \quad \left(\lambda \vec{x}. a \left[\bar{n} / \vec{x}(0) \right] \left[\vec{x} \downarrow \right] \right) \vec{p}$$

$$\left(\lambda \vec{x}. a \right) \iota \quad \mapsto_r \quad a \downarrow \vec{x}$$

extended to be applied under any context.

Example:

$$\begin{aligned} & \left(\lambda \vec{x}. \vec{x}(1) \left(\left[\lambda \vec{y}. \vec{x}(1) \iota \right] :: \iota \right) \right) \left([] :: [\textcolor{blue}{m}, \textcolor{blue}{m}] :: \iota \right) \\ & \mapsto_r \quad \left(\lambda \vec{x}. \textcolor{red}{x}(0) \left(\left[\lambda \vec{y}. \textcolor{red}{x}(0) \iota \right] :: \iota \right) \right) \left([\textcolor{blue}{m}, \textcolor{blue}{m}] :: \iota \right) \\ & \mapsto_r \quad 2 \cdot \left(\lambda \vec{x}. \textcolor{blue}{m} \left(\left[\lambda \vec{y}. \textcolor{blue}{m} \iota \right] :: \iota \right) \right) \iota \end{aligned}$$

Full step reduction

We define the **resource substitution** $q[\vec{n}/\vec{x}]$ of a **stream** $\vec{n}$ for a **stream variable** $\vec{x}$ in a resource expression q .

Example:

$$\begin{aligned} & \left(\vec{x}(1) \left([\lambda \vec{y}. \vec{x}(1) \iota, \lambda \vec{z}. \vec{x}(2) \iota] :: \iota \right) \right) [[] :: [m, m] :: [n] :: \iota / \vec{x}] \\ = & \quad 2 \cdot m \left([\lambda \vec{y}. m \iota, \lambda \vec{z}. n \iota] :: \iota \right) \end{aligned}$$

The **full-step reduction** is given by the basic rule:

$$(\lambda \vec{x}. a) \vec{n} \mapsto_R a [\vec{n} / \vec{x}]$$

extended to be applied under any context.

Properties of the extensional resource calculus

The fine-step resource reduction is extended to sums with $U \rightarrow_r V$ if:

- $U = \sum_{i=1}^k u_i$
- $V = \sum_{i=1}^k V_i$
- $\forall 1 \leq i \leq k, u_i \mapsto_r^? V_i$

Properties of $\rightarrow_r$.

The reduction $\rightarrow_r$ has the diamond property and is weakly normalizing.

Properties of the extensional resource calculus

The fine-step resource reduction is extended to sums with $U \rightarrow_r V$ if:

- $U = \sum_{i=1}^k u_i$
- $V = \sum_{i=1}^k V_i$
- $\forall 1 \leq i \leq k, u_i \mapsto_r^? V_i$

Properties of $\rightarrow_r$.

The reduction $\rightarrow_r$ has the diamond property and is weakly normalizing.

The full-step resource reduction is extended to sums with $U \rightarrow_R V$ if:

- $U = \sum_{i=0}^k u_i$
- $V = \sum_{i=0}^k V_i$
- $\forall 1 \leq i \leq k, u_i \mapsto_R^? V_i$ and $u_0 \mapsto_R V_0$

Properties of $\rightarrow_R$.

The reduction $\rightarrow_R$ is confluent and strongly normalizing.

Properties of the extensional resource calculus

The fine-step resource reduction is extended to sums with $U \rightarrow_r V$ if:

- $U = \sum_{i=1}^k u_i$
- $V = \sum_{i=1}^k V_i$
- $\forall 1 \leq i \leq k, u_i \mapsto_r^? V_i$

The full-step resource reduction is extended to sums with $U \rightarrow_R V$ if:

- $U = \sum_{i=0}^k u_i$
- $V = \sum_{i=0}^k V_i$
- $\forall 1 \leq i \leq k, u_i \mapsto_R^? V_i$ and $u_0 \mapsto_R V_0$

Properties of $\rightarrow_r$.

The reduction $\rightarrow_r$ has the diamond property and is weakly normalizing.

Properties of $\rightarrow_R$.

The reduction $\rightarrow_R$ is confluent and strongly normalizing.

Compatibility of the two reduction relations.

If $U \rightarrow_R V$, then $U \rightarrow_r^* V$.

Vectors of resource terms and promotion

A **value vector** is a vector M of value terms such that $\mathcal{V}_S(M) := \cup_{m \in M} \mathcal{V}_S(m)$ is finite. Likewise, we define **base vectors**, **bag vectors** and **stream vectors**.

Vectors of resource terms and promotion

A **value vector** is a vector M of value terms such that $\mathcal{V}_S(M) := \cup_{m \in M} \mathcal{V}_S(m)$ is finite. Likewise, we define **base vectors**, **bag vectors** and **stream vectors**.

The **promotion of a value vector** M , noted $M^!$, is defined with:

$$M^k := \sum_{\langle m_1, \dots, m_k \rangle \in \Delta_{\downarrow}^k} \left(\prod_{1 \leq i \leq k} M.m_i \right) [m_1, \dots, m_k] \quad \text{for } k \in \mathbb{N}$$
$$M^! := \sum_{k \in \mathbb{N}} \frac{1}{k!} M^k$$

Vectors of resource terms and promotion

A **value vector** is a vector M of value terms such that $\mathcal{V}_S(M) := \cup_{m \in M} \mathcal{V}_S(m)$ is finite. Likewise, we define **base vectors**, **bag vectors** and **stream vectors**.

The **promotion** of a value vector M , noted $M^!$, is defined with:

$$M^k := \sum_{\langle m_1, \dots, m_k \rangle \in \Delta_{\downarrow}^k} \left(\prod_{1 \leq i \leq k} M.m_i \right) [m_1, \dots, m_k] \quad \text{for } k \in \mathbb{N}$$
$$M^! := \sum_{k \in \mathbb{N}} \frac{1}{k!} M^k$$

If $\vec{M} := \langle M_i \rangle_{i \in \mathbb{N}}$ is a sequence of value vectors, its **promotion** is the stream vector $\vec{M}^!$ such that:

$$\text{for all } \vec{m} := \langle \bar{m}_i \rangle_{i \in \mathbb{N}} \in \Delta_S, \quad (\vec{M}^!).\vec{m} := \prod_{i \in \mathbb{N}} M_i^!. \bar{m}_i$$

Extensional Taylor Expansion

Step 1: Variable expansion.

Given a value variable x and a stream variable $\vec{x}$, we define simultaneously:

The **value expansion** of x is $x^\eta = \lambda \vec{y}. x \vec{y}^\dagger$ with $\vec{y}$ such that $x \notin \vec{y}$

The **stream expansion** of $\vec{x}$ is $\vec{x}^\dagger = (\langle \vec{x}(i)^\eta \rangle_{i \in \mathbb{N}})^\dagger$

Extensional Taylor Expansion

Step 1: Variable expansion.

Given a value variable x and a stream variable $\vec{x}$, we define simultaneously:

The **value expansion** of x is $x^\eta = \lambda \vec{y}. x \vec{y}^\dagger$ with $\vec{y}$ such that $x \notin \vec{y}$

The **stream expansion** of $\vec{x}$ is $\vec{x}^\dagger = (\langle \vec{x}(i)^\eta \rangle_{i \in \mathbb{N}})^\dagger$

Step 2: Structural Expansion.

The **structural Taylor expansion** is defined inductively as:

$$\mathcal{T}_\eta(x) := x^\eta \quad \mathcal{T}_\eta(\lambda x.M) := \lambda x. \mathcal{T}_\eta(M) \quad \mathcal{T}_\eta(M \ N) := \lambda \vec{y}. \mathcal{T}_\eta(M) \ \mathcal{T}_\eta(N)^\dagger :: \vec{y}^\dagger$$

where $\lambda x. \lambda \vec{y}. a := \lambda \vec{y}. a[\vec{y} \uparrow \{ \vec{y}(0)/x \}]$.

Resource reduction on term vectors

The **resource reduction on term vectors** is defined by $U \rightsquigarrow V$ when

- $U = \sum_{i \in I} \alpha_i u_i$ where the u_i 's are resource terms,
- $V = \sum_{i \in I} \alpha_i V_i$ where the V_i 's are sums of resource terms,
- $\forall i \in I, u_i \rightarrow_r^* V_i$.

Resource reduction on term vectors

The **resource reduction on term vectors** is defined by $U \rightsquigarrow V$ when

- $U = \sum_{i \in I} \alpha_i u_i$ where the u_i 's are resource terms,
- $V = \sum_{i \in I} \alpha_i V_i$ where the V_i 's are sums of resource terms,
- $\forall i \in I, u_i \rightarrow_r^* V_i$.

Lemma 3.

$$x^\eta \{m/x\} \rightsquigarrow m$$

$$m \{x^\eta/x\} \rightsquigarrow m$$

Lemma 4.

If $M \rightarrow_\eta M'$ then

$$\mathcal{T}_\eta(M) \rightsquigarrow \mathcal{T}_\eta(M')$$

Lemma 5.

If $M \rightarrow_\beta M'$ then

$$\mathcal{T}_\eta(M) \rightsquigarrow^* \mathcal{T}_\eta(M')$$

The λ -theory $=_{\mathcal{T}_\eta}$

We set the relation $=_{\mathcal{T}_\eta}$ with:

$$M =_{\mathcal{T}_\eta} N \quad \text{iff} \quad \text{NF}(\mathcal{T}_\eta(M)) = \text{NF}(\mathcal{T}_\eta(N))$$

Theorem 6: $=_{\mathcal{T}_\eta}$ is an extensional λ -theory.

$$\text{If } M =_{\beta\eta} M', \text{ then } \text{NF}(\mathcal{T}_\eta(M)) = \text{NF}(\mathcal{T}_\eta(M'))$$

The λ -theory $=_{\mathcal{T}_\eta}$

We set the relation $=_{\mathcal{T}_\eta}$ with:

$$M =_{\mathcal{T}_\eta} N \quad \text{iff} \quad \text{NF}(\mathcal{T}_\eta(M)) = \text{NF}(\mathcal{T}_\eta(N))$$

Theorem 6: $=_{\mathcal{T}_\eta}$ is an extensional λ -theory.

$$\text{If } M =_{\beta\eta} M', \text{ then } \text{NF}(\mathcal{T}_\eta(M)) = \text{NF}(\mathcal{T}_\eta(M'))$$

The theory $\mathbf{H}^*$ is the maximum consistent and sensible λ -theory.

Theorem 7.

The theory $=_{\mathcal{T}_\eta}$ is $\mathbf{H}^*$.

- I – Towards an Extensional λ -theory
- II – Extensional Resource Calculus
- III – Game Semantics: Pointer Concurrent Games

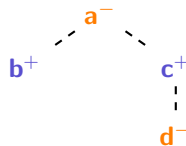
Arenas

An **arena** is a forest $A = \langle |A|, \rightarrow_A, \text{pol}_A \rangle$ with:

- $\text{pol}_A: |A| \rightarrow \{-, +\}$
- $\rightarrow_A$ alternating.

It is **pointed** if it has a single minimal event.

Example:



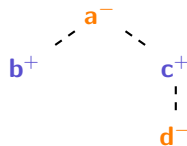
Arenas

An **arena** is a forest $A = \langle |A|, \rightarrow_A, \text{pol}_A \rangle$ with:

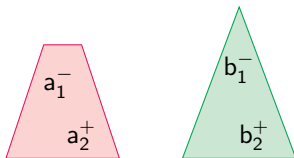
- $\text{pol}_A: |A| \rightarrow \{-, +\}$
- $\rightarrow_A$ alternating.

It is **pointed** if it has a single minimal event.

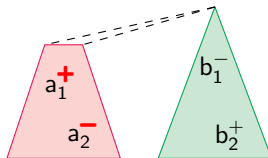
Example:



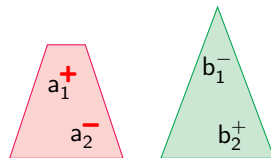
Constructions on arenas:



Arena $A \otimes B$.

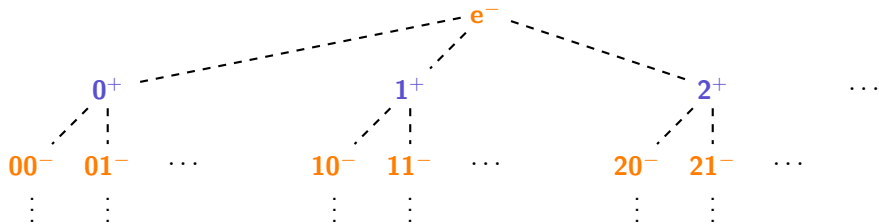


Arena $A \Rightarrow B$.



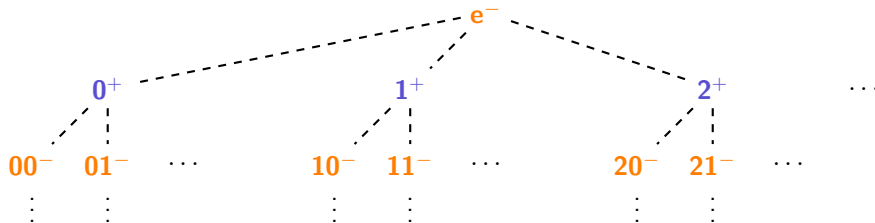
Arena $A \vdash B$.

The universal arena



The universal arena U

The universal arena



The universal arena $\mathbf{U}$

Useful isomorphisms.

$$\mathbf{U} \cong \mathbf{U} \Rightarrow \mathbf{U}$$

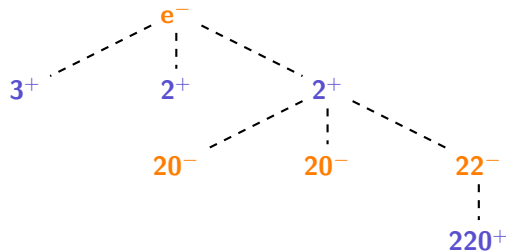
and

$$\mathbf{U} \cong \mathbf{U}^{\mathbb{N}} \Rightarrow o \quad \text{with} \quad \mathbf{U}^{\mathbb{N}} := \bigotimes_{n \in \mathbb{N}} \mathbf{U}$$

Positions and Augmentations

A **position** $x \in \text{Pos}(A)$ on an arena A is a thick subtree of A .
It is **pointed** if it has a single minimal event.

Example:

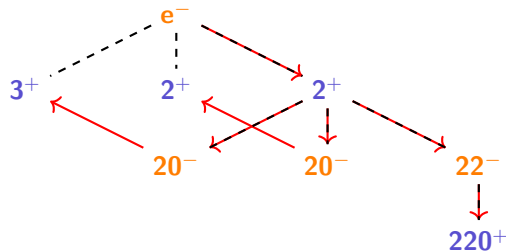


Positions and Augmentations

A **position** $x \in \text{Pos}(A)$ on an arena A is a thick subtree of A .
It is **pointed** if it has a single minimal event.

An **augmentation** $q \in \text{Aug}(A)$ is a position enriched with causal links.
It is **pointed** if it has a single minimal event.

Example:

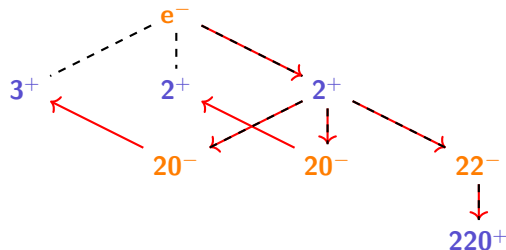


Positions and Augmentations

A **position** $x \in \text{Pos}(A)$ on an arena A is a thick subtree of A .
It is **pointed** if it has a single minimal event.

An **augmentation** $q \in \text{Aug}(A)$ is a position enriched with causal links.
It is **pointed** if it has a single minimal event.

Example:



This augmentation corresponds to the extensional resource term:

$$\lambda \vec{x}. \vec{x}(2) [\lambda \vec{y}. \vec{x}(3) \iota, \lambda \vec{y}. \vec{x}(2) \iota] :: [] :: [\lambda \vec{y}. \vec{y}(0) \iota] :: \iota$$

Normal extensional resource terms are augmentations

We obtain four mutually inductive bijections:

$$\| - \|_b^\Gamma : \Delta_b^{\text{NF}}(\Gamma) \rightarrow \text{Aug}_\bullet(\|\Gamma\| \vdash o)$$

$$\| - \|_v^\Gamma : \Delta_v^{\text{NF}}(\Gamma) \rightarrow \text{Aug}_\bullet(\|\Gamma\| \vdash \mathbf{U})$$

$$\| - \|_!^\Gamma : \Delta_!^{\text{NF}}(\Gamma) \rightarrow \text{Aug}(\|\Gamma\| \vdash \mathbf{U})$$

$$\| - \|_s^\Gamma : \Delta_s^{\text{NF}}(\Gamma) \rightarrow \text{Aug}(\|\Gamma\| \vdash \mathbf{U}^{\mathbb{N}})$$

Normal extensional resource terms are augmentations

Untyped setting:

We obtain four mutually inductive bijections:

$$\begin{aligned} \parallel - \parallel_b^\Gamma &: \Delta_b^{\text{NF}}(\Gamma) \rightarrow \text{Aug}_\bullet(\parallel \Gamma \parallel \vdash o) \\ \parallel - \parallel_v^\Gamma &: \Delta_v^{\text{NF}}(\Gamma) \rightarrow \text{Aug}_\bullet(\parallel \Gamma \parallel \vdash \mathbf{U}) \\ \parallel - \parallel_!^\Gamma &: \Delta_!^{\text{NF}}(\Gamma) \rightarrow \text{Aug}(\parallel \Gamma \parallel \vdash \mathbf{U}) \\ \parallel - \parallel_s^\Gamma &: \Delta_s^{\text{NF}}(\Gamma) \rightarrow \text{Aug}(\parallel \Gamma \parallel \vdash \mathbf{U}^{\mathbb{N}}) \end{aligned}$$

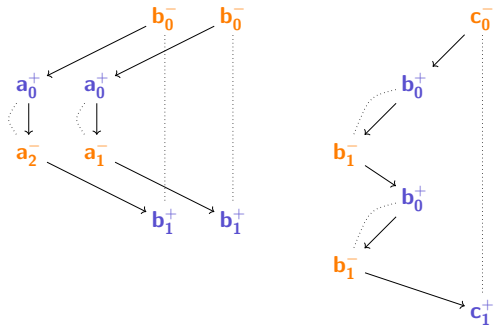
Typed setting:

We obtain similar bijections:

$$\begin{aligned} \parallel - \parallel_{\text{Tm}} &: \text{Tm}_{\text{NF}}(\Gamma; A) \cong \text{Aug}_\bullet(\parallel \Gamma \parallel \vdash \llbracket A \rrbracket) \\ \parallel - \parallel_{\text{Bg}} &: \text{Bg}_{\text{NF}}(\Gamma; A) \cong \text{Aug}(\parallel \Gamma \parallel \vdash \llbracket A \rrbracket) \\ \parallel - \parallel_{\text{Sq}} &: \text{Sq}_{\text{NF}}(\Gamma; \vec{A}) \cong \text{Aug}(\parallel \Gamma \parallel \vdash \llbracket \vec{A} \rrbracket) \end{aligned}$$

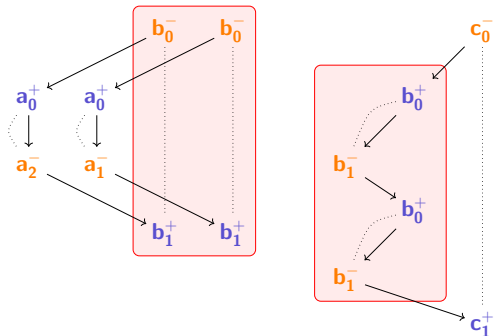
Composition

How do we compose $p: A \vdash B$ with $r: B \vdash C$?



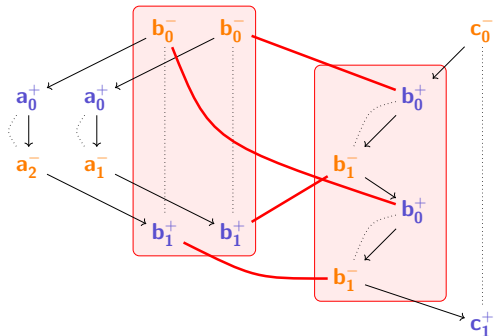
Composition

How do we compose $p: A \vdash B$ with $r: B \vdash C$?



Composition

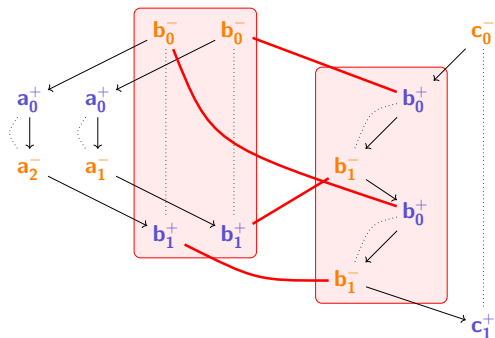
How do we compose $p: A \vdash B$ with $r: B \vdash C$?



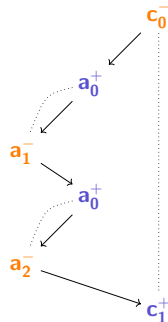
Interaction $r \circledast_{\varphi} p$

Composition

How do we compose $p: A \vdash B$ with $r: B \vdash C$?



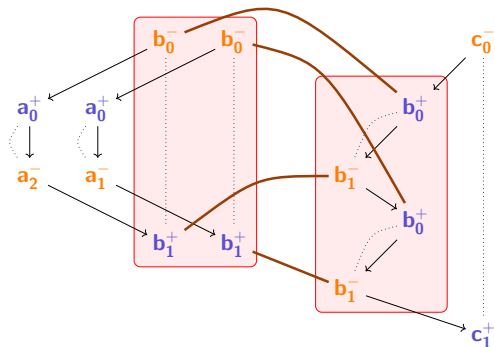
Interaction $r \otimes_{\varphi} p$



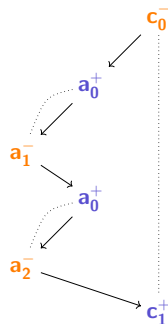
Composition $r \odot_{\varphi} p$

Composition

How do we compose $p: A \vdash B$ with $r: B \vdash C$?



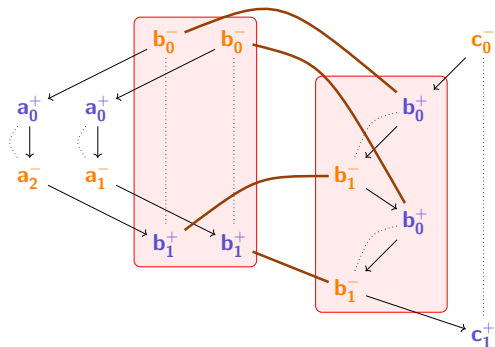
Interaction $r \circledast_{\psi} p$



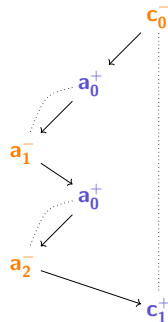
Composition $r \odot_{\varphi} p$

Composition

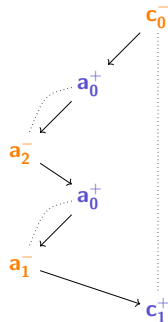
How do we compose $p: A \vdash B$ with $r: B \vdash C$?



Interaction $r \otimes_{\psi} p$



Composition $r \odot_{\varphi} p$



Composition $r \odot_{\psi} p$

Composition in PCG.

$$r \odot_{\varphi} p \neq r \odot_{\psi} p !$$

Composition and substitution

Composition in PCG.

$$r \odot_{\varphi} p \neq r \odot_{\psi} p !$$

The **composition** of r and p is:

$$r \odot p \stackrel{\text{def}}{=} \sum_{\varphi: r_{rhs} \cong p_{lhs}} r \odot_{\varphi} p$$

Composition and substitution

Composition in PCG.

$$r \odot_{\varphi} p \neq r \odot_{\psi} p !$$

The **composition** of r and p is:

$$r \odot p \stackrel{\text{def}}{=} \sum_{\varphi: r_{\text{rhs}} \cong p_{\text{lhs}}} r \odot_{\varphi} p$$

Substitution in the resource calculus.

$$\begin{aligned} & (\lambda \vec{y}. \textcolor{red}{x} ([\lambda \vec{z}. \textcolor{red}{x} \iota] :: \iota)) [[\textcolor{blue}{m}, \textcolor{violet}{n}]/\textcolor{red}{x}] \\ &= \lambda \vec{y}. \textcolor{blue}{m} ([\lambda \vec{z}. \textcolor{violet}{n} \iota] :: \iota) \\ &+ \lambda \vec{y}. \textcolor{violet}{n} ([\lambda \vec{z}. \textcolor{blue}{m} \iota] :: \iota) \end{aligned}$$

Composition and substitution

Composition in PCG.

$$r \odot_{\varphi} p \neq r \odot_{\psi} p !$$

The **composition** of r and p is:

$$r \odot p \stackrel{\text{def}}{=} \sum_{\varphi: r_{\text{rhs}} \cong p_{\text{lhs}}} r \odot_{\varphi} p$$

Substitution in the resource calculus.

$$\begin{aligned} & (\lambda \vec{y}. x \ ([\lambda \vec{z}. x \ \iota] :: \iota)) \ [[m, n]/x] \\ &= \lambda \vec{y}. m \ ([\lambda \vec{z}. n \ \iota] :: \iota) \\ &+ \lambda \vec{y}. n \ ([\lambda \vec{z}. m \ \iota] :: \iota) \end{aligned}$$

Intuitively, the substitution $m[\bar{n}/x]$ is:

$$m[\bar{n}/x] = \text{“} \sum_{\varphi: \substack{\text{occurrences} \\ \text{of } x \text{ in } m} \cong \bar{n}} m[\bar{n}/x]_{\varphi} \text{”}$$

Strategies in PCG are sums of augmentations:

$$\sigma = \sum_{p \in \text{Aug}(A)} \sigma(p) \cdot p$$

Strategies in PCG are sums of augmentations:

$$\sigma = \sum_{p \in \text{Aug}(A)} \sigma(p) \cdot p$$

The **composition** of the strategies $\sigma : A \vdash B$ and $\tau : B \vdash C$ is:

$$\tau \odot \sigma \stackrel{\text{def}}{=} \sum_{\substack{p \in \text{Aug}(A \vdash B) \\ r \in \text{Aug}(B \vdash C)}} \sigma(p) \tau(r) \cdot (r \odot p)$$

The *copycat* strategy

Consider $x \in \text{Pos}(A)$.

The **copycat augmentation** on x is:

$$cc_x = "x \vdash x"$$

The *copycat* strategy

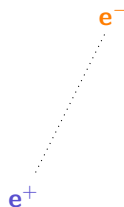
Consider $x \in \text{Pos}(A)$.

The **copycat augmentation** on x is:

$$cc_x = "x \vdash x"$$

Example:

$U \Rightarrow U$



The *copycat* strategy

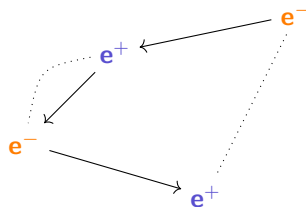
Consider $x \in \text{Pos}(A)$.

The **copycat augmentation** on x is:

$$cc_x = "x \vdash x"$$

Example:

$$U \Rightarrow U \vdash U \Rightarrow U$$



The *copycat* strategy

Consider $x \in \text{Pos}(A)$.

The **copycat augmentation** on x is:

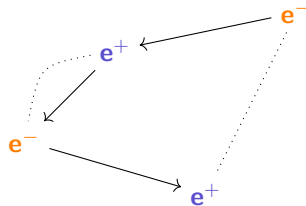
$$cc_x = "x \vdash x"$$

The **copycat strategy** on A is:

$$\text{id}_A \stackrel{\text{def}}{=} \sum_{x \in \text{Pos}(A)} \frac{1}{S(x)} \cdot cc_x$$

Example:

$$U \Rightarrow U \vdash U \Rightarrow U$$



PCG as a category

PCG is a category with:

- **Arenas** as objects;
- **Strategies** as morphisms;
- $\odot$ as the composition;
- **Copycats** as identities.

PCG as a category

PCG is a category with:

- **Arenas** as objects;
- **Strategies** as morphisms;
- $\odot$ as the composition;
- **Copycats** as identities.

Theorem 8.

PCG is a SMCC.

PCG as a category

PCG is a category with:

- **Arenas** as objects;
- **Strategies** as morphisms;
- $\odot$ as the composition;
- **Copycats** as identities.

Theorem 8.

PCG is a SMCC.

Theorem 9.

PCG is a closed **resource category**.

PCG as a category

PCG is a category with:

- **Arenas** as objects;
- **Strategies** as morphisms;
- $\odot$ as the composition;
- **Copycats** as identities.

Theorem 8.

PCG is a SMCC.

Theorem 9.

PCG is a closed **resource category**.

In the typed setting:

Theorem 10.

There is a sound interpretation

$$\llbracket - \rrbracket : \Delta \rightarrow \text{PCG}$$

Theorem 11.

For $s \in \text{tm}_{\text{NF}}(\Gamma; A)$, we have:

$$\llbracket s \rrbracket = 1 \cdot \|s\|_{\text{tm}}$$

Promotion

The **promotion** of a strategy $\sigma : \Gamma \vdash A$, noted $\sigma^!$, is defined with:

$$\sigma^k := \sum_{\bar{q} \in \text{Aug}(\Gamma \vdash A)^k} \left(\sum_{\langle q_1, \dots, q_k \rangle \in \text{enum}(\bar{q})} \prod_{1 \leq i \leq k} \sigma(q_i) \right) \cdot \bar{q} \quad \text{and} \quad \sigma^! := \sum_{k \in \mathbb{N}} \frac{1}{k!} \sigma^k$$

The **promotion** of a strategy $\sigma : \Gamma \vdash A$, noted $\sigma^!$, is defined with:

$$\sigma^k := \sum_{\bar{q} \in \text{Aug}(\Gamma \vdash A)^k} \left(\sum_{\langle q_1, \dots, q_k \rangle \in \text{enum}(\bar{q})} \prod_{1 \leq i \leq k} \sigma(q_i) \right) \cdot \bar{q} \quad \text{and} \quad \sigma^! := \sum_{k \in \mathbb{N}} \frac{1}{k!} \sigma^k$$

In the typed setting:

Theorem 11.

PCG is a closed resource category with promotion.

Theorem 12.

Consider $\Gamma \vdash M : F$. Then

$$\text{id}_{\llbracket F \rrbracket}^\bullet \odot \llbracket M \rrbracket = \llbracket \text{NF}(\mathcal{T}_\eta(M)) \rrbracket$$

Typed calculi:

$$\begin{array}{ccccc} M & \longrightarrow & \mathcal{T}_\eta(M) & \longrightarrow & \text{NF}(\mathcal{T}_\eta(M)) \\ \downarrow & & \downarrow & & \updownarrow \\ \llbracket M \rrbracket_\bullet & = & \llbracket \mathcal{T}(M) \rrbracket & = & \llbracket \text{NF}(\mathcal{T}(M)) \rrbracket \end{array}$$

**The normal form of the typed extensional
Taylor expansion is game semantics!**

Conclusion

Typed calculi:

$$\begin{array}{ccccc} M & \longrightarrow & \mathcal{T}_\eta(M) & \longrightarrow & \text{NF}(\mathcal{T}_\eta(M)) \\ \downarrow & & \downarrow & & \updownarrow \\ \llbracket M \rrbracket_\bullet & = & \llbracket \mathcal{T}(M) \rrbracket & = & \llbracket \text{NF}(\mathcal{T}(M)) \rrbracket \end{array}$$

The normal form of the typed extensional Taylor expansion is game semantics!

Untyped calculi:

- The relation $=_{\mathcal{T}_\eta}$ is an extensional λ -theory – and it's H^* .
- Normal extensional resource terms are in bijection with augmentations in the universal arena.

Conclusion

Typed calculi:

$$\begin{array}{ccccc} M & \longrightarrow & \mathcal{T}_\eta(M) & \longrightarrow & \text{NF}(\mathcal{T}_\eta(M)) \\ \downarrow & & \downarrow & & \updownarrow \\ \llbracket M \rrbracket_\bullet & = & \llbracket \mathcal{T}(M) \rrbracket & = & \llbracket \text{NF}(\mathcal{T}(M)) \rrbracket \end{array}$$

The normal form of the typed extensional Taylor expansion is game semantics!

Untyped calculi:

- The relation $=_{\mathcal{T}_\eta}$ is an extensional λ -theory – and it's H^* .
- Normal extensional resource terms are in bijection with augmentations in the universal arena.

Thank you for your attention!